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On the Elastic Equilibrium of Circular Cylinders under Certain Practical Systems of Load

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IV. *On the Elastic Equilibrium of Circular Cylinders under Certain Practical Systems of Load.*

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§ 1. *Object and Aims of the Paper.*

The usual solution for the extension and compression of elastic bars assumes that the latter are strained under a normal tension or pressure uniformly distributed across the plane ends. In like manner the solution for torsion of such bars assumes that the external forces which cause the torsion consist of a determinate system of tangential stresses, acting across the plane ends.

In both cases the solution is such that the torsion and extension are transmitted throughout the bar *without change of type.* Such terminal conditions of stress, however, do not usually occur in practice, and it accordingly becomes of considerable interest to find out how the results obtained for such a theoretical system of loading are modified, if at all, when we consider applied external stresses which give a closer representation of every-day mechanical conditions.

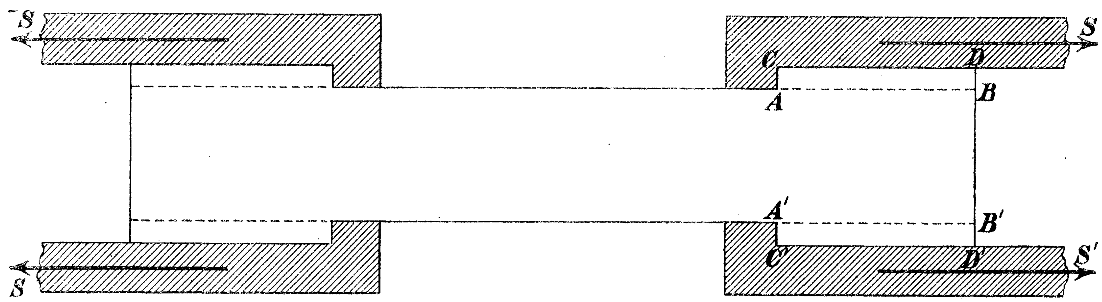
The present paper is an attempt towards the solution of this problem in three cases, which appear of especial practical interest.

The first case is that of a bar which is subjected to a determinate system of normal radial pressures and of axial shears all over the curved surface, the radial pressures being symmetrical about the mid-section and the shears having their sign changed. Thus the cylinder is subjected to a total axial pull, due to the shears, and also to a given transverse pressure. The plane ends are free from stress, except for a self-equilibrating system of radial shears, which will have little or no effect at points at some distance from the ends.

A special case worked out is that where the normal pressure is zero throughout, but a determinate axial shear, which has been taken constant, is made to act over two equal rings on the surface of the cylinder.

This will give us valuable information about a system of stress which often occurs in practice, in testing machines, for example, in which a specimen is pulled apart by means of pressures applied to the inner rims of projecting collars (see fig. 1). The shaded parts of the figure represent the "grips," and if S be the total pull applied, this is transmitted to the test piece by means of pressure applied along CA, C′A. Now consider the thinner cylinder in the middle ideally produced inside the thicker ends. It is in equilibrium under the stresses, radial and tangential, between the inner core and the hollow cylinder produced by the revolution of ABDC.

Fig. 1.



But what are these radial and tangential stresses? If we consider the equilibrium of the outer hollow cylinder only, we see that the resultant of the stresses across AB, A′B′ must exactly balance the pull S, however applied. The radial stress will probably be small, as it has no external traction to balance, and the longitudinal shears are therefore equivalent to S. Thus the thin cylinder inside is really stretched, not by normal traction over the flat ends, but by longitudinal shears over the curved surface, and a careful investigation will show that, in every *practical* case, extension is obtained by the application of an axial shear to the curved surface of the cylinder, *never* of tractions to the flat ends. The general effects of such a distribution appear, therefore, of great practical interest.

The second problem discussed is that of a cylinder of moderate length, which is compressed between two rough rigid planes in such a way that the terminal cross-sections are constrained to remain plane, but are not allowed to expand, their perimeter being kept fixed. By adding a suitable uniform distribution of pressure to a load system of this type, we can obtain the solution for a cylinder constrained in such a way that its ends expand by a definite amount. These two problems are of importance with reference to the behaviour of a block of stone or masonry when tested between millboard or metal planes, which practically hinder the block from expanding, and when tested between sheets of lead, which, on the other hand, favour the expansion of the block. The widely divergent results obtained for the strength of the same material when tested by these two methods have troubled many elasticians. UNWIN ('Testing of Materials of Construction') is of opinion that the results obtained when sheet lead is used are unreliable; whereas Professor PERRY, in his 'Applied Mechanics,' states that the true strength of the material is the one given by the lead experiments, and should be usually taken as half the published strength.

Finally, the third problem treated is that of a cylinder subjected to transverse shears over the parts of the curved surface near the ends, these shears being equivalent to a torsion couple. This is really the analogue, for torsion, of the first problem for tension, and corresponds to the case of a bar "gripped" as explained above, and twisted. This again is the method by which torsion is practically produced in most cases—almost always in laboratory experiments.

§ 2. *Method of Solution Adopted. Historical References.*

The method adopted has been to obtain symmetrical solutions of the equations of elasticity in cylindrical co-ordinates and to express the typical term in the form

$$\frac{\cos}{\sin}\{kz\} \times f(r),$$

r, ϕ, z being the usual cylindrical co-ordinates.

The expressions for the strains and stresses, over any coaxial cylinder, are therefere series of sines and cosines of multiples of z. The arbitrary constants of the coefficients are determined by comparison with the coefficients of the FOURIER'S series which express the applied stresses at the external boundary.

This method is not a new one. It has been indicated by LAMÉ and CLAPEYRON ("Mémoire sur l'équilibre intérieur des corps solides homogènes," 'Crelle's Journal,' vol. 7), but it has been for the first time worked out with any completeness by Professor L. POCHHAMMER ("Beitrag zur Theorie der Biegung des Kreiscylinders," 'Crelle's Journal,' vol. 81, 1876). Professor POCHHAMMER obtains the general solution of the elastic equations for an infinite circular cylinder subject to any system of surface loading, repeated at regular intervals. This he applies to the case of a built-in beam. The solution is not restricted to be symmetrical about the axis of the cylinder, but is perfectly general. The complete accurate expressions are, however, quite unwieldy; but, as the result of expanding the functions involved to the first two or three terms, Professor POCHHAMMER obtains far more manageable expressions, which he is eventually able to identify with those previously given by NAVIER and DE SAINT VENANT for more special cases of loading. It is to be noted, however, that POCHHAMMER restricts himself solely to the case of bending, and that his approximations depend upon the ratio of diameter to length being a small quantity.

The same general expressions have been independently arrived at by Mr. C. CHREE ("The Equations of an Isotropic Elastic Solid in Polar and Cylindrical Co-ordinates, their Solution and Application," 'Camb. Phil. Trans.,' vol. 14). Here, again, the solutions are not restricted to be symmetrical. The symmetrical terms, however, agree with the solutions of the present paper, but the latter are obtained by a process slightly different from that of Mr. CHREE. Mr. CHREE has also given a solution of the symmetrical case proceeding in powers of r and z. Using each form of solution *independently*, it is not possible to satisfy the condition that there shall be no stress at all on the curved surface; this is effected in the second problem of this paper, by means of a combination of the two types of solution.

In the paper referred to, Mr. CHREE, like Professor POCHHAMMER, has not, so far as I am aware, applied his general solution to the problems of tension and compression. He does give one example of torsion, which he obtains by applying an arbitrary system of cross-radial shears across the flat ends. Such a system, we have seen, would not usually correspond to what occurs in practice.

Mr. CHREE has written several other papers ("On some Compound Vibrating Systems," 'Camb. Phil. Trans.,' vol. 15, Part II.; "On Longitudinal Vibrations," 'Quarterly Journal of Mathematics,' 1889; "Longitudinal Vibrations in Solid and Hollow Cylinders," 'Phil. Mag.,' 1899; "On Long Rotating Circular Cylinders," 'Camb. Phil. Soc. Proc.,' vol. 7, Part VI., &c.), which deal with the solutions of the equations of elasticity in cylindrical co-ordinates, with special application to vibrations and rotating shafts; but I cannot find that he has anywhere returned to the statical problem and its solution by means of sine and cosine expansions.

[*October* 3, 1901.—Professor SCHIFF ('Journal de Liouville,' Série 3, vol. 9, 1883) has attempted the solution of the problem of the cylinder compressed between parallel planes, which is one of those treated of in the present paper. His solution is expressed in a series, not of circular functions, but of hyperbolic sines and cosines of nz, the successive values of n being obtained as roots of a certain transcendental equation. This enables him to satisfy the conditions at the curved surface, but the arbitrary coefficients are finally determined by the conditions over the plane ends. He assumes both the radial shear and the molecular rotation in a diametral plane to be given by known functions, $f(r)$ and $F(r)$, over the plane ends, and from these he succeeds in obtaining the coefficients. As he has only a single set of the latter left to carry out the identification, his functions $f(r)$ and $F(r)$ are not really independent. Theoretically only the shear $f(r)$ should be required, and in a practical problem even this is unknown, the total pressure being all that is given. The actual distribution of this pressure does not appear to enter into Professor SCHIFF's solution. Also the fact that the values of n are roots of a transcendental equation singularly complicates the solution from a numerical point of view, and Professor SCHIFF appears to have made no attempt to translate his results into numbers.]*

It has therefore appeared worth while to apply the solutions involving circular functions of z to problems such as those sketched above.

Of each of these I have given a concrete numerical example. Indeed, the greater part of the work has been spent on these numerical examples. The labour of calculation has in most cases been considerable, owing to the slow convergence of many of the series involved, which has necessitated special methods of approximation.

* Since writing the above, I find that the problem of the circular cylinder under a symmetrical strain has been considered by J. THOMAE in two papers ("Über eine einfache Aufgabe aus der Theorie der Elasticität," 'Leipzig Berichte,' vols. 37–38). The author has used expansions in sines and cosines of kz, but, as far as I can make out, the only problem he considers is that of the vertical pillar under its own weight.

§ 3. *General Solution for a Symmetrical Strain.*

Let r, ϕ, z be the usual cylindrical co-ordinates; also, following the notation of TODHUNTER and PEARSON's 'History of Elasticity,' let $\widehat{st}$ denote the stress, parallel to ds, across an element of surface perpendicular to dt, s, t standing for any two of the letters r, ϕ, z.

Let u, v, w denote the radial, cross-radial, and longitudinal displacements respectively, then we have (LAMÉ, 'Leçons sur l'Elasticité'), if u, v, w are independent of ϕ:

$$(\lambda + 2\mu)\frac{d^2u}{dr^2} + (\lambda + 2\mu)\frac{d}{dr}\left(\frac{u}{r}\right) + \mu\frac{d^2u}{dz^2} + (\lambda + \mu)\frac{d^2w}{drdz} = 0 \quad . \quad . \quad . \quad (1).$$

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d(rv)}{dr}\right) + \frac{d^2v}{dz^2} = 0 \quad . \quad . \quad . \quad (2).$$

$$(\lambda + \mu)\left(\frac{d^2u}{drdz} + \frac{1}{r}\frac{du}{dz}\right) + \mu\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) + (\lambda + 2\mu)\frac{d^2w}{dz^2} = 0 \quad . \quad . \quad . \quad (3).$$

$$\left.\begin{aligned} \widehat{rr} &= (\lambda + 2\mu)\frac{du}{dr} + \lambda\frac{u}{r} + \lambda\frac{dw}{dz} & \widehat{rz} &= \mu\left(\frac{dw}{dr} + \frac{du}{dz}\right) \\ \widehat{zz} &= \lambda\frac{du}{dr} + \lambda\frac{u}{r} + (\lambda + 2\mu)\frac{dw}{dz} & \widehat{\phi z} &= \mu\frac{dv}{dz} \\ \widehat{\phi\phi} &= (\lambda + 2\mu)\frac{u}{r} + \lambda\frac{du}{dr} + \lambda\frac{dw}{dz} & \widehat{r\phi} &= \mu\left(\frac{dv}{dr} - \frac{v}{r}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (4),$$

λ and μ being the elastic constants of LAMÉ.

We see from the above that $\widehat{\phi z}$ and $\widehat{r\phi}$ depend only on v, the other stresses only on u and w. Also the equation (2) contains v only, (1) and (3) contain u and w only. The solution for transverse displacements is therefore absolutely independent of the solution for radial and longitudinal displacements.

Let us now denote the operators $\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r$ by ϑ^2, and $\frac{d}{dz}$ by D. Differentiate (1) with regard to z and (3) with regard to r, and remember that the order of the symbols D and ϑ is indifferent, then we find

$$(\vartheta^2 + D^2)\,v = 0 \quad . \quad . \quad . \quad (5).$$

$$\left((\lambda + 2\mu)\,\vartheta^2 + \mu D^2\right)\frac{du}{dz} + (\lambda + \mu)D^2\frac{dw}{dr} = 0 \quad . \quad . \quad . \quad (6).$$

$$(\lambda + \mu)\vartheta^2\frac{du}{dz} + \left(\mu\vartheta^2 + (\lambda + 2\mu)\,D^2\right)\frac{dw}{dr} = 0 \quad . \quad . \quad . \quad (7).$$

Eliminate either du/dz or dw/dr between (6) and (7): it is found that either of these quantities satisfies the partial differential equation

$$(\vartheta^2 + D^2)^2 y = 0 \quad \ldots \ldots \ldots \quad (8).$$

The whole problem of the determination of the elastical equilibrium of the circular cylinder under any symmetrical system of stress depends therefore on the solution of this differential equation.

§ 4. *Solution of the Differential Equation.*

The differential equation

$$(\vartheta^2 + D^2)\, y = 0$$

is really identical with LAPLACE's equation in cylindrical co-ordinates, namely,

$$\frac{1}{r}\frac{d}{dr}\frac{rdV}{dr} + \frac{1}{r^2}\frac{d^2V}{d\phi^2} + \frac{d^2V}{dz^2} = 0.$$

Suppose V independent of ϕ and differentiate with regard to r, we have

$$(\vartheta^2 + D^2)\, dV/dr = 0.$$

If therefore V be such a solution of LAPLACE's equation, $y = dV/dr$ will be a solution of the given differential equation. For our purpose, however, it will be simpler to proceed from the equation itself.

Assume a typical solution

$$y_1 = R_1 \,.\, Z_1,$$

where R_1 is a function of r only, Z_1 a function of z only.

We find easily

$$\frac{d^2R_1}{dr^2} + \frac{1}{r}\frac{dR_1}{dr} - \left(\frac{1}{r^2} + k^2\right)R_1 = 0 \quad \ldots \ldots \ldots \quad (9),$$

$$\frac{d^2Z_1}{dz^2} + k^2 Z_1 = 0 \quad \ldots \ldots \ldots \quad (10).$$

The solutions of (9) are of the form

$$I_1(kr) \quad \text{and} \quad K_1(kr),$$

where

$$I_n(x) = \sum_0^\infty \frac{x^{n+2s}}{2^{n+2s}\Pi(s)\,\Pi(s+n)},$$

$$K_n(x) = (-1)^n \frac{1\,.\,3\,.(2n-1)}{x^n}\int_0^\infty \frac{\cos(x\sinh\phi)}{\cosh^{2n}\phi}\,d\phi\,.$$

(See GRAY and MATHEWS, 'Bessel's Functions,' pp. 66–7.)

Consider now the equation

$$(\vartheta^2 + D^2)^2 y = 0.$$

Let y_1 be any solution of the equation

$$(\vartheta^2 + D^2)\, y = 0.$$

Then if y_2 be a solution of

$$(\vartheta^2 + D^2)\, y = y_1$$

it is also a solution of

$$(\vartheta^2 + D^2)^2 y = 0.$$

Now if $y_1 = R_1Z_1$ what is the condition that we can obtain a second product solution $y_2 = R_2Z_2$?

We have, substituting

$$\frac{\vartheta^2 R_2}{R_2} + \frac{D^2 Z_2}{Z_2} = \frac{R_1}{R_2} \cdot \frac{Z_1}{Z_2} \quad \ldots \ldots \ldots \quad (11),$$

or function of r only + function of z only = product function in r and z.

If (11) is to be identically satisfied, this product function must be a function of r only or of z only.

Case (i.). $Z_2 = aZ_1$ where a is a constant.

We find

$$\vartheta^2 R_2 - k^2 R_2 = \frac{1}{a} R_1 \quad \ldots \ldots \ldots \quad (12),$$

or, since R_1 is a solution of

$$(\vartheta^2 - k^2)\, R_1 = 0 \quad \ldots \ldots \ldots \quad (13),$$

which is the same as (9), R_2 is a solution of

$$(\vartheta^2 - k^2)^2 R_2 = 0,$$

which is not at the same time a solution of (13).

Now the solutions of this equation are

$$I_1(kr), \quad K_1(kr), \quad \frac{d}{dk} I_1(kr), \quad \frac{d}{dk} K_1(kr).$$

But

$$\frac{d}{dk} I_1(kr) = r I_1'(kr) = r I_0(kr) - \frac{1}{k} I_1(kr),$$

and similarly

$$\frac{d}{dk} K_1(kr) = r K_0(kr) - \frac{1}{k} K_1(kr).$$

The four independent integrals are therefore

$$I_1(kr), \quad K_1(kr), \quad r I_0(kr), \quad r K_0(kr),$$

and therefore the required values of R_2 are rI_0, rK_0.

Case (ii.). $R_2 = bR_1$; we find, using (13),

$$(D^2 + k^2) Z_2 = \frac{1}{b} Z_1,$$

and therefore Z_2 is a solution of $(D^2 + k^2)^2 Z_2 = 0$, which is not at the same time a solution of

$$(D^2 + k^2) Z_2 = 0.$$

The possible values of Z_2 are $z \cos kz$, $z \sin kz$.
Hence the possible sets of product functions satisfying the equation

$$(\vartheta^2 + D^2)^2 y = 0$$

are as follows:—

$$\left.\begin{aligned} y = &\ A \cos(kz + \alpha) I_1(kr) \\ &\ B \cos(kz + \beta) K_1(kr) \\ &\ C \cos(kz + \gamma) rI_0(kr) \\ &\ D \cos(kz + \delta) rK_0(kr) \\ &\ Ez \cos(kz + \epsilon) I_1(kr) \\ &\ Fz \cos(kz + \theta) K_1(kr) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

§ 5. *Solution under given conditions of Surface-loading; the first problem.*

Let us now consider first the case of a circular cylinder under the following system of stress:

$\widehat{rr}/\mu$ = a given even function of z $(= f(z))$ over the curved surface $r = a$,

$\widehat{rz}/\mu$ = a given odd function of z $(= \psi(z))$ over the curved surface $r = a$,

$\widehat{zz} = 0$ over the plane ends $z = \pm c$.

Since du/dz, dw/dr are both solutions of (8) we may have them composed of a series of terms as follows:

$$\frac{du}{dz} = \Sigma \left\{ \begin{aligned} A_1 \cos(kz + \alpha_1) I_1(kr) + C_1 \cos(kz + \gamma_1) rI_0(kr) \\ + E_1 z \cos(kz + \epsilon_1) I_1(kr) \end{aligned} \right\} \quad . \quad . \quad . \quad (15).$$

$$\frac{dw}{dr} = \Sigma \left\{ \begin{aligned} A_2 \cos(kz + \alpha_2) I_1(kr) + C_2 \cos(kz + \gamma_2) rI_0(kr) \\ + E_2 z \cos(kz + \epsilon_2) I_1(kr) \end{aligned} \right\} \quad . \quad . \quad . \quad (16).$$

No K-functions have been introduced in this case, as they lead to infinite terms at the axis.

Also the conditions of the problem require that u shall be an even function of z and w an odd function of z. Hence

$$\alpha_1 = \alpha_2 = \gamma_1 = \gamma_2 = -\pi/2, \ \epsilon_1 = \epsilon_2 = 0.$$

Integrating (15) and (16) we have

$$u = \chi(r) + \Sigma\left\{-\frac{A_1}{k}\cos kz I_1(kr) - \frac{C_1}{k}\cos kz . rI_0(kr) + \frac{E_1}{k}\left(z\sin kz + \frac{\cos kz}{k}\right)I_1(kr)\right\},$$

$$w = \theta(z) + \Sigma\left\{\frac{A_2}{k}\sin kz I_0(kr) + \frac{C_2}{k}\sin kz . rI_1(kr) + \frac{E_2}{k}z\cos kz\, I_0(kr)\right\}.$$

To find the relations between the constants we must substitute in equations (1) and (3). We then find the following relations:

$$\vartheta^2\chi(r) = 0 \quad . \quad . \quad . \quad . \quad (17), \qquad D^2\theta(z) = 0 \quad . \quad . \quad . \quad . \quad (18),$$

$$(A_1 - A_2)(\lambda + \mu)k^2 + 2k\{C_1(\lambda + 2\mu) - \mu E_1 - (\lambda + \mu)E_2\} = 0 \quad . \quad . \quad (19),$$

$$(A_1 - A_2)(\lambda + \mu)k^2 + 2k\{C_1(\lambda + \mu) + \mu C_2 - (\lambda + 2\mu)E_2\} = 0 \quad . \quad . \quad (20),$$

$$C_1 = C_2 = C, \text{ say} \quad . \quad . \quad . \quad (21), \qquad E_1 = E_2 = E, \text{ say} \quad . \quad . \quad . \quad (22).$$

In virtue of equations (21) and (22), (19) and (20) reduce to the single equation

$$(A_1 - A_2)(\lambda + \mu)k + 2(\lambda + 2\mu)(C - E) = 0 \quad . \quad . \quad . \quad . \quad . \quad (23).$$

Also from (17) and (18)

$$\chi(r) = u_0 r, \qquad \theta(z) = w_0 z,$$

remembering that w is odd in z and that u is not to be infinite when $r = 0$.

For the stresses $\widehat{rr}$, $\widehat{zz}$, $\widehat{rz}$ we find from (4), after some obvious reductions,

$$\begin{aligned}\widehat{rr} = {} & 2(\lambda + \mu)u_0 + \lambda w_0 \\ & + \Sigma\left[\begin{array}{l}\dfrac{-\mu}{\lambda + 2\mu}\{(2\lambda + 3\mu)A_1 + \mu A_2\}I_0(kr)\cos kz \\ + 2\mu\left\{\left(A_1 - \dfrac{E}{k}\right)\dfrac{I_1(kr)}{kr}\cos kz + Ez\sin kz\left(I_0(kr) - \dfrac{I_1(kr)}{kr}\right)\right\} - CrI_1(kr)\cos kz\end{array}\right]\end{aligned} \quad . \quad . \quad . \quad (24).$$

$$\begin{aligned}\widehat{zz} = {} & 2\lambda u_0 + (\lambda + 2\mu)w_0 \\ & + \Sigma\left[\begin{array}{l}\left\{(\lambda + 2\mu)A_2 - \lambda A_1 - \lambda\dfrac{2C}{k} + 2(\lambda + \mu)\dfrac{E}{k}\right\}I_0(kr)\cos kz \\ \quad + 2\mu\{CrI_1(kr)\cos kz - EI_0(kr)z\sin kz\}\end{array}\right]\end{aligned} \quad . \quad . \quad . \quad . \quad (25).$$

$$\widehat{rz} = \mu\Sigma\{(A_1 + A_2)I_1(kr)\sin kz + 2CrI_0(kr)\sin kz + 2EI_1(kr)z\cos kz\} \quad . \quad . \quad . \quad (26).$$

But clearly if $\widehat{zz}$ is to be zero all over the plane ends we must have

$$u_0 = -\frac{\lambda + 2\mu}{2\lambda} w_0 \quad \ldots \quad (27),$$

$$k = \frac{(2n+1)\pi}{2c} \quad \ldots \quad (28),$$

and

$$E = 0 \quad \ldots \quad (29).$$

The expressions for the displacements and stresses then reduce to the following, writing $kr = \rho$ for shortness :

$$u = u_0 r - \Sigma \left\{ A_1 I_1(\rho) + \frac{C}{k} \rho I_0(\rho) \right\} \frac{\cos kz}{k} \quad \ldots \quad (30),$$

$$w = w_0 z + \Sigma \left\{ A_2 I_0(\rho) + \frac{C}{k} \rho I_1(\rho) \right\} \frac{\sin kz}{k} \quad \ldots \quad (31),$$

$$\widehat{rr} = 2(\lambda + \mu) u_0 + \lambda w_0$$
$$+ \Sigma \left\{ -\frac{\mu}{\lambda + 2\mu} [(2\lambda + 3\mu) A_1 + \mu A_2] I_0(\rho) + 2\mu \left[\frac{A_1}{\rho} - \frac{C\rho}{k} \right] I_1(\rho) \right\} \cos kz. \quad (32),$$

$$\widehat{zz} = \Sigma \left\{ \frac{\mu}{\lambda + 2\mu} [(3\lambda + 4\mu) A_2 - \lambda A_1] I_0(\rho) + 2\mu \frac{C}{k} \rho I_1(\rho) \right\} \cos kz \quad \ldots \quad (33),$$

$$\widehat{rz} = \mu \Sigma \left\{ (A_1 + A_2) I_1(\rho) + \frac{2C}{k} \rho I_0(\rho) \right\} \sin kz \quad \ldots \quad (34),$$

where

$$\frac{2C}{k} = -\frac{\lambda + \mu}{\lambda + 2\mu} (A_1 - A_2) \quad \ldots \quad (35).$$

Over the surface of the cylinder $r = a$, we find

$$(\widehat{rr}/\mu)_{r=a} = -\frac{3\lambda + 2\mu}{\lambda} w_0$$
$$+ \Sigma \left\{ \begin{array}{l} A_1 \left[-(1+\gamma) I_0(\alpha) + \left(\frac{2}{\alpha} + \gamma\alpha \right) I_1(\alpha) \right] \\ + A_2 [-(1-\gamma) I_0(\alpha) - \gamma\alpha I_1(\alpha)] \end{array} \right\} \cos kz \quad \ldots \quad (36),$$

$$(\widehat{rz}/\mu)_{r=a} = \Sigma \left\{ \begin{array}{l} A_1 [I_1(\alpha) - \gamma\alpha I_0(\alpha)] \\ + A_2 [I_1(\alpha) + \gamma\alpha I_0(\alpha)] \end{array} \right\} \sin kz \quad \ldots \quad (37),$$

where γ is written for $\dfrac{\lambda + \mu}{\lambda + 2\mu}$ and α for ka.

Now $f(z)$ being an even function, we can expand it in a FOURIER'S series between the limits $\pm c$ in the form

$$\left.\begin{aligned} f(z) - f(c) &= \sum_0^\infty a_n \cos \frac{\widehat{2n+1}\pi z}{2c} \\ \text{where} \qquad a_n &= \frac{1}{c}\int_{-c}^{+c} \{f(z) - f(c)\} \cos \frac{\widehat{2n+1}\pi z}{2c}\, dz \end{aligned}\right\} \quad \ldots\ldots \quad (38),$$

and $\psi(z)$ being an odd function can be expanded in the form

$$\left.\begin{aligned} \psi(z) &= \sum_0^\infty b_n \sin \frac{\widehat{2n+1}\pi z}{2c} \\ \text{where} \qquad b_n &= \frac{1}{c}\int_{-c}^{+c} \psi(z) \sin \frac{\widehat{2n+1}\pi z}{2c}\, dz \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (39).$$

Now since $\qquad (\widehat{rr})_{r=a} = \mu f(z),\ (\widehat{rz})_{r=a} = \mu\psi(z),$

we have, comparing (36) and (37) with (38) and (39),

$$-\frac{3\lambda + 2\mu}{\lambda} w_0 = f(c)$$

$$A_1\left(-\{1+\gamma\}I_0(\alpha) + \left\{\frac{2}{\alpha} + \gamma\alpha\right\}I_1(\alpha)\right) + A_2(-\{1-\gamma\}I_0(\alpha) - \gamma\alpha I_1(\alpha)) = a_n$$

$$A_1(I_1(\alpha) - \gamma\alpha I_0(\alpha)) + A_2(I_1(\alpha) + \gamma\alpha I_0(\alpha)) = b_n,$$

whence

$$A_1 = -\tfrac{1}{2}\frac{a_n(\alpha I_1(\alpha) + \gamma\alpha^2 I_0(\alpha)) + b_n(\gamma\alpha^2 I_1(\alpha) + (1-\gamma)\alpha I_0(\alpha))}{\gamma\alpha^2 I_0^2(\alpha) - (1+\gamma\alpha^2) I_1^2(\alpha)} \quad \ldots \quad (40),$$

$$A_2 = \tfrac{1}{2}\frac{a_n(\alpha I_1(\alpha) - \gamma\alpha^2 I_0(\alpha)) + b_n((1+\gamma)\alpha I_0(\alpha) - (2+\gamma\alpha^2) I_1(\alpha))}{\gamma\alpha^2 I_0^2(\alpha) - (1+\gamma\alpha^2) I_1^2(\alpha)} \quad \ldots \quad (41),$$

$$w_0 = -\frac{(2\gamma - 1)}{4\gamma - 1} f(c) \quad \ldots\ldots\ldots \quad (42),$$

and therefore from (35) and (27)

$$C = \frac{(2n+1)\pi}{2c}\,\tfrac{1}{2}\frac{a_n\alpha\gamma I_1(\alpha) + b_n(\gamma\alpha I_0(\alpha) - \gamma I_1(\alpha))}{\gamma\alpha^2 I_0^2(\alpha) - (1+\gamma\alpha^2) I_1^2(\alpha)} \quad \ldots\ldots \quad (43),$$

$$u_0 = \frac{1}{2(4\gamma - 1)} f(c) \quad \ldots\ldots\ldots \quad (44).$$

§ 6. *Consideration of the Approximate Expressions to which the Results of the last Section lead, when the Ratio of Diameter to Length is small.*

If we can treat the diameter of the cylinder as small compared with its length, we can obtain a first approximation by following the method of Professor POCHHAMMER ('Crelle,' vol. 81), and expanding A_1, A_2, C in powers of α, which is then a small quantity, provided the index n is not too large.* If we do this we find

$$A_1 = -a_n \frac{2\gamma+1}{4\gamma-1} + \frac{b_n}{\alpha}\frac{2\gamma-2}{4\gamma-1},$$

$$A_2 = -a_n \frac{2\gamma-1}{4\gamma-1} + \frac{b_n}{\alpha}\frac{2\gamma}{4\gamma-1},$$

$$\frac{C}{k} = a_n \frac{\gamma}{4\gamma-1} + \frac{b_n}{\alpha}\frac{\gamma}{4\gamma-1},$$

and, expanding $I_0(kr)$ and $I_1(kr)$ in powers of r, and dropping all the terms except the first (which is really equivalent to a second approximation, since the indices go up two at a time), we find

$$u = \frac{r}{2(4\gamma-1)}\left\{f(c) + \Sigma\left(a_n - \frac{4\gamma-2}{\alpha}b_n\right)\cos kz\right\}$$

$$= \frac{r}{2(4\gamma-1)}\left\{f(c) + f(z) - f(c) + \frac{4\gamma-2}{a}\int_c^z \psi(z)\,dz\right\},$$

using the Fourier expansions (38) and (39)

$$= \frac{r}{4\gamma-1}\frac{1}{2\mu}\left\{(\widehat{rr})_{r=a} - (2\gamma-1)\frac{1}{\pi a^2}\int_z^c (\widehat{rz})_{r=a}\,dz \times 2\pi a\right\}.$$

Now $\int_z^c (\widehat{rz})_{r=a}\,dz \times 2\pi a$ is equal to the total longitudinal pull exerted on the bar by all the forces on one side of the cross-section considered. It represents, in other words, the total tension at that cross-section. Denoting it by $\pi a^2 Q$, where Q is the mean tension at that cross-section,

$$u = r\left\{(\widehat{rr})_{r=a} \times \frac{\lambda+2\mu}{2\mu(3\lambda+2\mu)} - Q\frac{\lambda}{2\mu(3\lambda+2\mu)}\right\} \quad . \quad . \quad . \quad . \quad . \quad (45),$$

which shows that the radial displacement is exactly the same as if the only forces on a thin lamina between two cross-sections were an external radial tension $(\widehat{rr})_{r=a}$ and a uniform tension Q across the plane faces.

* For the analytical restrictions necessary in such a case, see § 28.

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In like manner it can be shown that

$$\left.\begin{aligned} w &= w_0 z + \Sigma\left(-a_n \frac{2\gamma-1}{4\gamma-1} + \frac{b_n}{\alpha}\frac{2\gamma}{4\gamma-1}\right)\frac{\sin kz}{k} \\ &= -\frac{2\gamma-1}{4\gamma-1}\left\{z f(c) + \int_0^z (f(z)-f(c))\,dz\right\} + \frac{2\gamma}{4\gamma-1}\frac{1}{\alpha}\int_z^c dz\int_0^z \psi(z)\,dz \\ &= -\frac{2\gamma-1}{4\gamma-1}\int_0^z \frac{(\widehat{rr})_{r=a}}{\mu}\,dz + \frac{\gamma}{4\gamma-1}\int_0^z \frac{Q}{\mu}\,dz \\ &= \int_0^z\left\{-\frac{\lambda}{3\lambda+2\mu}\frac{(\widehat{rr})_{r=a}}{\mu} + \frac{\lambda+\mu}{3\lambda+2\mu}\frac{Q}{\mu}\right\}dz \\ &= \int_0^z s_z dz \end{aligned}\right\} \quad (46),$$

s_z being the stretch parallel to the axis in a cylinder which is under a tension Q across its plane faces and a radial tension $(\widehat{rr})_{r=a}$.

Thus the longitudinal and radial displacements are, to a first approximation, the same as if the cylinder were supposed made up of any number of thin circular laminæ, piled up on top of each other, the longitudinal tension in any lamina being uniform and giving a total tension equal to the total pull of all the external forces acting on the cylinder on one side of the section considered.

Further, the shearing stress $\widehat{rz}$ at a point inside is found to the same approximation to be given by

$$\frac{\widehat{rz}}{\mu} = \Sigma\,(kr)\,\frac{b_n}{\alpha}\sin kz = \frac{r}{a}\Sigma b_n \sin kz = \frac{r}{a}\frac{(\widehat{rz})_{r=a}}{\mu}$$

so that, in the parts of the cylinder to which external shearing stress is applied, and in these only, there is shearing stress inside the cylinder, which shearing stress is proportional to the distance from the axis.

The other stresses, $\widehat{rr}$, $\widehat{\phi\phi}$, $\widehat{zz}$, are found to the same approximation to be all constants for any given value of z.

$$\widehat{zz} = 2\mu\Sigma\frac{b_n}{\alpha}\cos kz = \frac{2\mu}{a}\int_z^c \psi(z)\,dz = Q,$$

$$\widehat{rr} = \widehat{\phi\phi} = \mu f(z).$$

It follows from the above that the action of any radial pressure will be purely local, and also that, whatever the manner in which the cylinder is "gripped" and the pull is applied, the stress in the portions of the bar between the points of application of the pull reduces practically to a uniform tension.

The above results are somewhat remarkable as tending to show how very restricted is the effect of local stresses, provided they leave no total resultant, and how, when they do leave a total resultant, the effect of this resultant is practically independent of the manner in which it is applied. This is the celebrated "principle of the equivalence of statically equipollent loads," which was first enunciated by DE SAINT-VENANT on general physical principles, and has been considerably confirmed by BOUSSINESQ's researches on the effect of small local surface actions.

It is to be borne in mind, of course, that the solution obtained in § 5, although making $\widehat{zz} = 0$ over the flat ends, does not at the same time ensure $\widehat{rz} = 0$. In other words, we have a determinate system of radial shears over the flat ends, but from symmetry this system must be self-equilibrating. The disturbances due to it will therefore, by the above principle, be purely local, and, provided we remove the ends sufficiently far from the parts of the beam which we desire to study, no trouble need arise on account of all the conditions not being strictly satisfied.

§ 7. *Numerical Problem. Expressions for Strains and Stresses.*

Let us now return to the exact expressions and apply them to the case of a comparatively short cylinder.

Suppose that $\widehat{rr} = 0$ all over the curved surface and that in some way, as described in § 1, a shear $\widehat{rz}$, which we shall take uniform and equal to S, is made to act along two rings upon the curved surface, so that

$$(\widehat{rz})_{r=a} = 0 \text{ when } -b+e < z < b-e$$

$$z < -b-e, \quad z > b+e$$

$$(\widehat{rz})_{r=a} = \mathrm{S} \text{ when } b-e < z < b+e$$

$$(\widehat{rz})_{r=a} = -\mathrm{S} \text{ when } -b-e < z < -b+e.$$

We have then

$$u_0 = w_0 = 0,$$

$$a_n = 0,$$

$$\mu b_n = \frac{8\mathrm{S}}{(2n+1)\pi} \sin \frac{\overline{2n+1}\pi e}{2c} \sin \frac{\overline{2n+1}\pi b}{2c}.$$

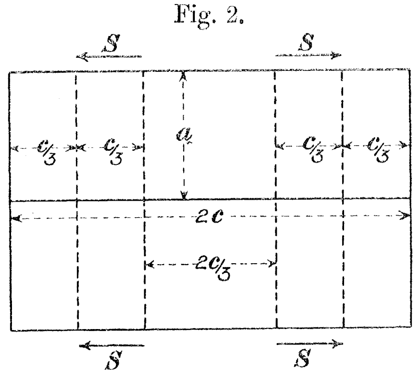
The expressions for the constants, stresses, and displacements then reduce to

$$\left.\begin{aligned}
\mathrm{A}_1 &= -\frac{4\mathrm{S}}{(2n+1)\pi\mu} \sin \frac{\overline{2n+1}\pi e}{2c} \sin \frac{\overline{2n+1}\pi b}{2c} \frac{\gamma\alpha^2\mathrm{I}_1 + (1-\gamma)\alpha\mathrm{I}_0}{\gamma\alpha^2\mathrm{I}_0^2 - (1+\gamma\alpha^2)\mathrm{I}_1^2} \\
\mathrm{A}_2 &= \frac{4\mathrm{S}}{(2n+1)\pi\mu} \sin \frac{\overline{2n+1}\pi e}{2c} \sin \frac{\overline{2n+1}\pi b}{2c} \frac{(1+\gamma)\alpha\mathrm{I}_0 - (2+\gamma\alpha^2)\mathrm{I}_1}{\gamma\alpha^2\mathrm{I}_0^2 - (1+\gamma\alpha^2)\mathrm{I}_1^2} \\
\frac{\mathrm{C}}{k} &= \frac{4\mathrm{S}}{(2n+1)\pi\mu} \sin \frac{\overline{2n+1}\pi e}{2c} \sin \frac{\overline{2n+1}\pi b}{2c} \frac{\gamma\alpha\mathrm{I}_0 - \gamma\mathrm{I}_1}{\gamma\alpha^2\mathrm{I}_0^2 - (1+\gamma\alpha^2)\mathrm{I}_1^2}
\end{aligned}\right\} \quad . \quad . \quad (47).$$

$$\left.\begin{aligned}
\frac{\widehat{rr}}{\mu} &= \Sigma\left[-\left\{(1+\gamma)\,\mathrm{A}_1 + (1-\gamma)\,\mathrm{A}_2\right\}\mathrm{I}_0(\rho) + 2\left(\frac{\mathrm{A}_1}{\rho} - \frac{\mathrm{C}}{k}\rho\right)\mathrm{I}_1(\rho)\right]\cos kz \\
\frac{\widehat{\phi\phi}}{\mu} &= \Sigma\left[-\frac{2\mathrm{A}_1}{\rho}\mathrm{I}_1(\rho) - \frac{1-\gamma}{\gamma}\frac{2\mathrm{C}}{k}\mathrm{I}_0(\rho)\right]\cos kz \\
\frac{\widehat{zz}}{\mu} &= \Sigma\left[\left\{(2\gamma+1)\,\mathrm{A}_2 - (2\gamma-1)\,\mathrm{A}_1\right\}\mathrm{I}_0(\rho) + \frac{2\mathrm{C}}{k}\rho\mathrm{I}_1(\rho)\right]\cos kz \\
\frac{\widehat{rz}}{\mu} &= \Sigma\left[(\mathrm{A}_1+\mathrm{A}_2)\,\mathrm{I}_1(\rho) + \frac{2\mathrm{C}}{k}\rho\mathrm{I}_0(\rho)\right]\sin kz \\
u &= -\Sigma\left(\mathrm{A}_1\mathrm{I}_1(\rho) + \frac{\mathrm{C}}{k}\rho\mathrm{I}_0(\rho)\right)\frac{\cos kz}{k} \\
w &= \Sigma\left(\mathrm{A}_2\mathrm{I}_0(\rho) + \frac{\mathrm{C}}{k}\rho\mathrm{I}_1(\rho)\right)\frac{\sin kz}{k}
\end{aligned}\right\} . \quad (48).$$

In the above α is the argument of the I-functions, unless the argument is written. To simplify the expressions we shall take $\pi a = 2c$, so that the length is about three times the radius. This makes $\alpha = 2n + 1$ $(n = 0, 1, 2, \ldots)$. Further, suppose $e = c/6$, $b = c/2$, so that the cylinder is divided into 5 zones, as shown in fig. 2.

Fig. 2.



The middle one from $-c/3$ to $+c/3$, unstressed; two rings from $c/3$ to $+2c/3$ and $-c/3$ to $-2\,c/3$ over which a uniform shear is acting; finally, the outer rings $2\,c/3$ to c and $-2\,c/3$ to $-c$, which are unstressed. Also, in order to simplify still more, we shall suppose POISSON'S ratio to have the value 1/4, or $\gamma = 2/3$.

It may be objected, it is true, that in many actual materials POISSON'S ratio is not 1/4. But this is not really an objection, because the object of this investigation is not so much to find out the *absolute values* of strains and stresses in any given material, as to calculate the *alterations* in these values as deduced from the hypothesis of uniform stress, and this we can best do by taking a value for POISSON'S ratio which is, on the whole, well within the limits indicated by practical results, and which makes the arithmetic somewhat easier.

If we do this and calculate the values ot the constants, we find that for the first 10 terms they come to the following values :

TABLE of Constants.

$\frac{\mu\pi A_1}{4S}$.	$\frac{\mu\pi A_2}{4S}$.	$\frac{\mu\pi C}{4Sk}$.
− ·272602	+ ·205790	+ ·159464
− ·142172	− ·0359051	+ ·0354223
+ ·0323529	+ ·0163035	− ·00534980
+ ·00492450	+ ·00308747	− ·000612345
− ·000534505	− ·000374718	+ ·0000532623
− ·0000285886	− ·0000214593	+ ·00000237642
− ·00000413531	− ·00000325082	+ ·000000294829
− ·00000162159	− ·00000131798	+ ·000000101204
+ ·000000315996	+ ·000000263390	− ·0000000175353
+ ·0000000448505	+ ·0000000381292	− ·00000000224045

From these I have calculated the coefficients of the FOURIER's series for the stresses and strains for $r = 0$, $r = ·2a$, $r = ·4a$, and $r = ·6a$. For higher values of r the convergence becomes slower and the expressions more difficult to handle. In the case of the stresses and strains at the boundary $r = a$, special methods of approximation have to be resorted to.

The expressions for the strains and stresses are:

$$u = \frac{8Sc}{\mu\pi^2}\left[\begin{array}{l} -\ ·00482 \cos\frac{\pi z}{2c} + ·00380 \cos\frac{3\pi z}{2c} - ·00230 \cos\frac{5\pi z}{2c} \\ -\ ·00043 \cos\frac{7\pi z}{2c} + ·00006 \cos\frac{9\pi z}{2c} + \dots \end{array}\right] (r = ·2a),$$

$$u = \frac{8Sc}{\mu\pi^2}\left[\begin{array}{l} -\ ·01075 \cos\frac{\pi z}{2c} + ·01412 \cos\frac{3\pi z}{2c} - ·00541 \cos\frac{5\pi z}{2c} \\ -\ ·00130 \cos\frac{7\pi z}{2c} + ·00023 \cos\frac{9\pi z}{2c} + ·00002 \cos\frac{11\pi z}{2c} + \dots \end{array}\right] (r = ·4a),$$

$$u = \frac{8Sc}{\mu\pi^2}\left[\begin{array}{l} -\ ·01897 \cos\frac{\pi z}{2c} + ·02014 \cos\frac{3\pi z}{2c} - ·00991 \cos\frac{5\pi z}{2c} \\ -\ ·00330 \cos\frac{7\pi z}{2c} + ·00084 \cos\frac{9\pi z}{2c} + ·00011 \cos\frac{11\pi z}{2c} \\ +\ ·00004 \cos\frac{13\pi z}{2c} + ·00005 \cos\frac{15\pi z}{2c} - ·00002 \cos\frac{17\pi z}{2c} \\ -\ ·00001 \cos\frac{19\pi z}{2c} + \dots \end{array}\right] (r = ·6a),$$

and in like manner for w and the stresses.

To save space, the coefficients of the series may be exhibited in tabular form as follows :—

Coefficient of	$\widehat{zz} \times \frac{\pi}{4S}$. $r = 0$.	$r = (\cdot 2)\,a$.	$r = (\cdot 4)\,a$.	$r = (\cdot 6)\,a$.	$\widehat{rr} \times \frac{\pi}{4S}$. $r = 0$.	$r = (\cdot 2)\,a$.	$r = (\cdot 4)\,a$.	$r = (\cdot 6)\,a$.
$\cos \pi z/2c$	+ ·57104	+ ·58318	+ ·62014	+ ·68364	+ ·11314	+ ·10923	+ ·09721	+ ·07616
$\cos 3\pi z/2c$	− ·03639	− ·02640	+ ·01004	+ ·09557	+ ·10675	+ ·10983	+ ·11683	+ ·11921
$\cos 5\pi z/2c$	+ ·02726	+ ·02846	+ ·02810	+ ·00614	− ·02700	− ·03253	− ·04981	− ·07754
$\cos 7\pi z/2c$	+ ·00556	+ ·00732	+ ·01181	+ ·01457	− ·00431	− ·00679	− ·01547	− ·03650
$\cos 9\pi z/2c$	− ·00070	− ·00113	− ·00298	− ·00692	+ ·00048	+ ·00099	+ ·00354	+ ·01242
$\cos 11\pi z/2c$	− ·00004	− ·00009	− ·00036	− ·00136	+ ·00003	+ ·00007	+ ·00040	+ ·00209
$\cos 13\pi z/2c$	− ·00001	− ·00002	− ·00011	− ·00068	—	+ ·00002	+ ·00013	+ ·00096
$\cos 15\pi z/2c$	—	− ·00001	− ·00010	− ·00089	—	+ ·00001	+ ·00011	+ ·00119
$\cos 17\pi z/2c$	—	—	+ ·00004	+ ·00058	—	—	− ·00004	− ·00074
$\cos 19\pi z/2c$	—	—	+ ·00001	+ ·00027	—	—	− ·00001	− ·00033

Coefficient of	$\widehat{\phi\phi} \times \pi/4S$. $r = 0$.	$r = (\cdot 2)\,a$.	$r = (\cdot 4)\,a$.	$r = (\cdot 6)\,a$.
$\cos \pi z/2c$	+ ·11314	+ ·11291	+ ·11218	+ ·11091
$\cos 3\pi z/2c$	+ ·10675	+ ·10998	+ ·11998	+ ·13760
$\cos 5\pi z/2c$	− ·02700	− ·02980	− ·03927	− ·05916
$\cos 7\pi z/2c$	− ·00431	− ·00528	− ·00907	− ·01922
$\cos 9\pi z/2c$	+ ·00048	+ ·00068	+ ·00159	+ ·00489
$\cos 11\pi z/2c$	+ ·00003	+ ·00004	+ ·00015	+ ·00065
$\cos 13\pi z/2c$	—	+ ·00001	+ ·00004	+ ·00025
$\cos 15\pi z/2c$	—	—	+ ·00003	+ ·00026
$\cos 17\pi z/2c$	—	—	− ·00001	− ·00014
$\cos 19\pi z/2c$	—	—	—	− ·00006

Coefficient of	$w \times \mu\pi^2/8Sc$. $r = 0$.	$r = (\cdot 2)\,a$.	$r = (\cdot 4)\,a$.	$r = (\cdot 6)\,a$.	$\widehat{rz} \times \pi/4S$. $r = (\cdot 2)\,a$.	$r = (\cdot 4)\,a$.	$r = (\cdot 6)\,a$.
$\sin \pi z/2c$	+ ·20579	+ ·21106	+ ·22712	+ ·25475	+ ·05771	+ ·11909	+ ·18801
$\sin 3\pi z/2c$	− ·01197	− ·01085	− ·00655	+ ·00418	− ·00944	− ·00878	+ ·01915
$\sin 5\pi z/2c$	+ ·00326	+ ·00352	+ ·00403	+ ·00323	+ ·01395	+ ·02861	+ ·03569
$\sin 7\pi z/2c$	+ ·00044	+ ·00058	+ ·00103	+ ·00163	+ ·00444	+ ·01219	+ ·02464
$\sin 9\pi z/2c$	− ·00004	− ·00007	− ·00019	− ·00050	− ·00082	− ·00310	− ·00955
$\sin 11\pi z/2c$	—	—	− ·00002	− ·00007	− ·00007	− ·00037	− ·00171
$\sin 13\pi z/2c$	—	—	− ·00001	− ·00003	− ·00002	− ·00012	− ·00081
$\sin 15\pi z/2c$	—	—	—	− ·00003	− ·00001	− ·00010	− ·00104
$\sin 17\pi z/2c$	—	—	—	+ ·00002	—	+ ·00004	+ ·00066
$\sin 19\pi z/2c$	—	—	—	+ ·00001	—	+ ·00001	+ ·00030

Using the above values I have calculated the stresses and strains for the points $r = 0$, $a/5$, $2a/5$, $3a/5$, and $z = \pm(0, c/10, 2c/10, \&c.)$. These are tabulated on pp. 171–173.

§ 8. *Calculation of the Stresses on the Outer Surface of the Cylinder.*

Along the outer surface $r = a$, $\rho = \alpha$, and we have the following expressions for the stresses $\widehat{\phi\phi}$ and $\widehat{zz}$, and the displacements u and w: $\widehat{rz}$ and $\widehat{rr}$ of course are known.

Consider for example the stresses $(\widehat{zz})_{r=a}$ and $(\widehat{\phi\phi})_{r=a}$.

They are

$$(\widehat{zz})_{r=a} = \frac{4S}{\pi} \sum_0^\infty \frac{6\gamma\alpha I_0^2 - (4\gamma + 2) I_0 I_1 - 2\gamma\alpha I_1^2}{(\gamma\alpha^2 I_0^2 - (1 + \gamma\alpha^2) I_1^2)(2n + 1)} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi b}{2c} \cos \frac{\widehat{2n+1}\pi z}{2c} \quad . \quad . \quad . \quad (49),$$

and

$$(\widehat{\phi\phi})_{r=a} = \frac{4S}{\pi} \sum_0^\infty \frac{2\gamma\alpha I_1^2 + 4(1-\gamma) I_0 I_1 - 2(1-\gamma)\alpha I_0^2}{(\gamma\alpha^2 I_0^2 - (1 + \gamma\alpha^2) I_1^2)(2n + 1)} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi b}{2c} \cos \frac{\widehat{2n+1}\pi z}{2c} \quad . \quad . \quad . \quad (50).$$

Now, when α is fairly large (say > 10), I_0 and I_1 may be replaced by their semi-convergent expansions:

$$I_0(\alpha) = \sqrt{\frac{1}{2\pi\alpha}}\, e^\alpha \left(1 + \frac{1^2}{8\alpha} + \frac{1^2 . 3^2}{2!(8\alpha)^2} + \frac{1^2 . 3^2 . 5^2}{3!(8\alpha)^3} + \ldots\right)$$

$$I_1(\alpha) = \sqrt{\frac{1}{2\pi\alpha}}\, e^\alpha \left(1 - \frac{3}{8\alpha} - \frac{3 . 5}{2!(8\alpha)^2} - \frac{3 . 5 . 21}{3!(8\alpha)^3} - \ldots\right)$$

(see GRAY and MATHEWS, 'Bessel's Functions,' p. 68). From which we find that the coefficients of $\cos \frac{\widehat{2n+1}\pi z}{2c}$ in the expansions of $\frac{\pi}{4S}(\widehat{\phi\phi})_{r=a}$ and $\frac{\pi}{4S}(\widehat{zz})_{r=a}$ approximate to the values (remembering $\alpha = 2n + 1$):

$$\left\{\frac{4\gamma - 2}{\gamma} \frac{1}{2n+1} + \frac{(2 - 2\gamma)(3\gamma - 1)}{\gamma^2} \frac{1}{(2n+1)^2}\right\} \sin \frac{\widehat{2n+1}\pi e}{2c} \times \sin \frac{\widehat{2n+1}\pi b}{2c}$$

and

$$\left\{\frac{4}{2n+1} + \frac{2(1-\gamma)}{\gamma} \frac{1}{(2n+1)^2}\right\} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi b}{2c}.$$

Now let us write

$$\frac{6\gamma\alpha I_0^2 - (4\gamma + 2) I_0 I_1 - 2\gamma\alpha I_1^2}{\gamma\alpha^2 I_0^2 - (1 + \gamma\alpha^2) I_1^2} \frac{1}{(2n+1)} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi b}{2c}$$

$$= \left\{\frac{4}{2n+1} + \frac{2(1-\gamma)}{\gamma} \frac{1}{(2n+1)^2}\right\} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi b}{2c} + q_n' \quad . \quad . \quad . \quad . \quad . \quad (51),$$

$$\frac{2\gamma\alpha I_1^2 + 4(1-\gamma) I_0 I_1 - 2(1-\gamma)\alpha I_0^2}{\gamma\alpha^2 I_0^2 - (1 + \gamma\alpha^2) I_1^2} \frac{1}{(2n+1)} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi b}{2c}$$

$$= \left\{\frac{4\gamma - 2}{\gamma} \frac{1}{2n+1} + \frac{(2-2\gamma)(3\gamma-1)}{\gamma^2} \frac{1}{(2n+1)^2}\right\} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi b}{2c} + p_n' \; . \; (52),$$

so that p_n', q_n' are comparable with the terms of the series $\Sigma \frac{1}{(2n+1)^3}$, which converge fairly rapidly. We see therefore that $\widehat{zz}$ and $\widehat{\phi\phi}$ are made up of two kinds of terms (a), terms of the form $\frac{4S}{\pi} \Sigma p_n' \cos \frac{\widehat{2n+1}\pi z}{2c}$ and $\frac{4S}{\pi} \Sigma q_n' \cos \frac{\widehat{2n+1}\pi z}{2c}$, which are absolutely and uniformly convergent series, and (b) series, in which the coefficients are the approximate expressions found above. Of the series (b), those which have terms containing $1/(2n+1)^2$ or $1/(2n+1)^3$ are absolutely and uniformly convergent. This, however, is not the case with the series formed by taking the leading terms in the approximation, viz.:—

$$\frac{4S}{\pi} \sum_0^\infty \frac{8}{(2n+1)} \sin \frac{(2n+1)\pi e}{2c} \sin \frac{(2n+1)\pi b}{2c} \cos \frac{(2n+1)\pi z}{2c}$$

and

$$\frac{4S}{\pi} \sum_0^\infty \frac{4\gamma - 2}{\gamma(2n+1)} \sin \frac{(2n+1)\pi e}{2c} \sin \frac{(2n+1)\pi b}{2c} \cos \frac{(2n+1)\pi z}{2c}.$$

For the series

$$\sum_0^\infty \frac{1}{(2n+1)} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi b}{2c} \cos \frac{\widehat{2n+1}\pi z}{2c}$$

may be broken up into the sum of four other series, thus:

$$\tfrac{1}{4} \sum_0^\infty \frac{1}{(2n+1)} \cos \frac{\widehat{2n+1}\pi}{2c} (z - b + e) + \tfrac{1}{4} \sum_0^\infty \frac{1}{(2n+1)} \cos \frac{\widehat{2n+1}\pi}{2c} (z + b - e)$$

$$- \tfrac{1}{4} \sum_0^\infty \frac{1}{(2n+1)} \cos \frac{\widehat{2n+1}\pi}{2c} (z + b + e) - \tfrac{1}{4} \sum_0^\infty \frac{1}{(2n+1)} \cos \frac{\widehat{2n+1}\pi}{2c} (z - b - e).$$

Now it is easy to show that

$$\sum_0^\infty \frac{1}{(2n+1)} \cos (2n+1) x = \tfrac{1}{2} \log \left(\cot \frac{|x|}{2}\right)$$

where $|x|$ is the numerical value of x.

The series on the left is divergent and $\log\left(\cot\frac{|x|}{2}\right) = \infty$ if $x = 0$. We see, therefore, that, at the points $z = \pm b \pm e$, *i.e.*, wherever the shear $\widehat{rz}$ changes discontinuously, the stresses $\widehat{zz}$ and $\widehat{\phi\phi}$ become infinite.

The meaning of this in practice would be that, as the transition from the stressed to the unstressed surface becomes more abrupt, the tractions in the neighbourhood become dangerously large. And if the shear is applied by means of a projecting rim or collar of material, on which the pull is brought to bear, as in fig. 1, then this rim or collar must not project out of the material at a sharp angle, or in any way which tends to introduce a discontinuous tangential stress over the surface of the cylinder. This is already recognised in practice; test pieces, which are thicker at the ends than in the middle, being made in such a way that the transition from the smaller to the larger diameter is gradual.

The series containing $1/(2n+1)^2$ can also be evaluated in finite terms:

$$\sum_0^\infty \frac{1}{(2n+1)^2} \sin\frac{\widehat{2n+1}\pi e}{2c} \sin\frac{\widehat{2n+1}\pi b}{2c} \cos\frac{\widehat{2n+1}\pi z}{2c}$$

$$= \frac{\pi}{2c}\int^c \sum_0^\infty \frac{1}{(2n+1)} \sin\frac{\widehat{2n+1}\pi e}{2c} \sin\frac{\widehat{2n+1}\pi b}{2c} \sin\frac{\widehat{2n+1}\pi z}{2c}$$

$$= \frac{\pi^2}{16cS}\int_z^c (\widehat{rz})_{r=a}\, dz$$

$$= 0 \text{ from } z = -c \text{ to } -b-e$$

$$\frac{\pi^2}{16c}(z+b+e) \text{ from } z = -b-e \text{ to } z = -b+e$$

$$\frac{\pi^2 e}{8c} \qquad \text{from } z = -b+e \text{ to } z = b-e$$

$$\frac{\pi^2}{16c}(b+e-z) \text{ from } z = b-e \text{ to } z = b+e$$

$$0 \qquad \text{from } z = b+e \text{ to } z = c.$$

Thus we have only to calculate p_n', q_n' and to sum the corresponding series, the rest of the expressions for the stresses being reducible to finite terms.

For $\gamma = 2/3$, I find the values of p_n', q_n' to be given by:

n	p_n'	q_n'
0	$-\cdot 35130$	$-\cdot 01184$
1	$-\cdot 04818$	$-\cdot 05960$
2	$+\cdot 00838$	$+\cdot 01431$
3	$+\cdot 00217$	$+\cdot 00436$
4	$-\cdot 00061$	$-\cdot 00135$
5	$-\cdot 00011$	$-\cdot 00025$
6	$-\cdot 00006$	$-\cdot 00015$
7	$-\cdot 00010$	$-\cdot 00026$
8	$+\cdot 00008$	$+\cdot 00024$
9	$+\cdot 00006$	$+\cdot 00018$

from which the values of $\widehat{zz}$, $\widehat{\phi\phi}$ can be found when $r = a$. They are tabulated, with the other stresses, upon p. 171.

§ 9. *Calculation of the Displacements on the Outer Surface of the Cylinder.*

In a precisely similar manner we find for the displacements

$$(u)_{r=a} = -\frac{8Sc}{\mu\pi^2}\sum_0^\infty \left\{1 - \frac{a\mathrm{I}_0/\mathrm{I}_1 - 1}{\gamma a^2 \mathrm{I}_0^2/\mathrm{I}_1^2 - 1 - \gamma a^2}\right\}\frac{\sin\dfrac{\widehat{2n+1}\pi e}{2c}}{(2n+1)^2}\sin\frac{\widehat{2n+1}\pi b}{2c}\cos\frac{\widehat{2n+1}\pi z}{2c}$$

$$= -\frac{8Sc}{\mu\pi^2}\sum_0^\infty \frac{1-\dfrac{1}{\gamma}}{(2n+1)^2}\sin\frac{\widehat{2n+1}\pi e}{2c}\sin\frac{\widehat{2n+1}\pi b}{2c}\cos\frac{\widehat{2n+1}\pi z}{2c}$$

$$-\frac{8Sc}{\mu\pi^2}\Sigma u_n'\cos\frac{\widehat{2n+1}\pi z}{2c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where

$$u_n' = \left(\frac{1}{\gamma} - \frac{a\mathrm{I}_0/\mathrm{I}_1 - 1}{\gamma a^2\mathrm{I}_0^2/\mathrm{I}_1^2 - 1 - \gamma a^2}\right)\frac{1}{(2n+1)^2}\sin\frac{\widehat{2n+1}\pi e}{2c}\sin\frac{\widehat{2n+1}\pi b}{2c}$$

and is of the order $1/(2n+1)^3$

$$(w)_{r=a} = \frac{8Sc}{\mu\pi^2}\sum_0^\infty \frac{(1+\gamma)a\mathrm{I}_0^2 - \gamma a\mathrm{I}_1^2 - 2\mathrm{I}_1\mathrm{I}_0}{(\gamma a^2\mathrm{I}_0^2 - (1+\gamma a^2)\mathrm{I}_1^2)(2n+1)^2}\sin\frac{\widehat{2n+1}\pi e}{2c}\sin\frac{\widehat{2n+1}\pi b}{2c}\sin\frac{\widehat{2n+1}\pi z}{2c}$$

$$= \frac{8Sc}{\mu\pi^2}\sum_0^\infty \frac{1}{\gamma}\frac{1}{(2n+1)^2}\sin\frac{(2n+1)\pi e}{2c}\sin\frac{\widehat{2n+1}\pi b}{2c}\sin\frac{\widehat{2n+1}\pi z}{2c}$$

$$+\frac{8Sc}{\mu\pi^2}\sum_0^\infty\left(\frac{1-\gamma}{\gamma}\right)^2\frac{1}{(2n+1)^3}\sin\frac{(2n+1)\pi e}{2c}\sin\frac{(2n+1)\pi b}{2c}\sin\frac{(2n-1)\pi z}{2c}$$

$$+\frac{8Sc}{\mu\pi^2}\sum_0^\infty w_n'\sin\frac{\widehat{2n+1}\pi z}{2c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54),$$

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where

$$w_n' = \left\{\frac{(1+\gamma)\,\alpha I_0^2 - \gamma\alpha I_1^2 - 2I_1I_0}{\gamma\alpha^2 I_0^2 - (1+\gamma\alpha^2)\,I_1^2} - \frac{1}{\gamma} - \left(\frac{1-\gamma}{\gamma}\right)^2\frac{1}{\alpha}\right\}\frac{1}{(2n+1)^2}\sin\frac{\overline{2n+1}\pi e}{2c}\sin\frac{\overline{2n+1}\pi b}{2c},$$

and is of order $1/(2n+1)^4$.

It so happens that in w the term of order $1/(2n+1)^3$ is evaluable in finite terms, and I have included it.

It is easy to see that

$$\sum_0^\infty \frac{1}{(2n+1)^3}\sin\frac{(2n+1)\pi e}{2c}\sin\frac{(2n+1)\pi b}{2c}\sin\frac{\overline{2n+1}\pi z}{2c}$$

$$= -\frac{\pi^3 eb}{16c^3} \text{ from } z = -c \text{ to } z = -b-e$$

$$= -\left\{2eb - \tfrac{1}{2}(b+e+z)^2\right\}\frac{\pi^3}{32c^2} \text{ from } z = -b-e \text{ to } z = -b+e$$

$$= \frac{\pi^3 ez}{16c^2} \text{ from } z = -b+e \text{ to } z = b-e$$

$$= \left\{2eb - \tfrac{1}{2}(b+e-z)^2\right\}\frac{\pi^3}{32c^2} \text{ from } z = b-e \text{ to } z = b+e$$

$$= \frac{\pi^3 eb}{16c^2} \text{ from } z = b+e \text{ to } z = +c.$$

The leading series in w cannot, however, be evaluated so easily. It is seen to depend upon the evaluation of the series

$$\sum_0 \frac{\sin\overline{2n+1}x}{(2n+1)^2} = \int_0^x \tfrac{1}{2}\log\left(\cot\frac{x}{2}\right)dx$$

$$= \frac{x}{2}\log\left(\cot\frac{x}{2}\right) + \int_0^x \frac{x}{2}\operatorname{cosec} x\,dx.$$

As series of this kind are frequently turning up in investigations like the present, I have tabulated below the values of $\tfrac{1}{2}\int_0^x \frac{x}{\sin x}\,dx$ and also of $\sum_0^\infty \frac{\sin\overline{2n+1}x}{(2n+1)^2}$ for values of x ranging from 0 to $\pi/2$ at intervals of $\pi/40$. Intermediate values are then obtained by interpolation when required.

TABLE of $\tfrac{1}{2}\int_0^x \frac{x}{\sin x}\,dx = f(x)$.

x.	$f(x)$.	x.	$f(x)$.	x.	$f(x)$.	x.	$f(x)$.
$\pi/40$	·039283	$6\pi/40$	·238572	$11\pi/40$	·450873	$16\pi/40$	·690354
$2\pi/40$	·078648	$7\pi/40$	·279605	$12\pi/40$	·496043	$17\pi/40$	·743248
$3\pi/40$	·118174	$8\pi/40$	·321246	$13\pi/40$	·542417	$18\pi/40$	·798291
$4\pi/40$	·157947	$9\pi/40$	·363596	$14\pi/40$	·590147	$19\pi/40$	·855760
$5\pi/40$	·198050	$10\pi/40$	·406766	$15\pi/40$	·639400	$20\pi/40$	·915963

TABLE of $\sum_0^\infty \frac{\sin \widehat{2n+1}x}{(2n+1)^2}$.

x.	Σ.	x.	Σ.	x.	Σ.	x.	Σ.
$\pi/40$	·16639	$6\pi/40$	·57475	$11\pi/40$	·78536	$16\pi/40$	·89109
$2\pi/40$	·27830	$7\pi/40$	·62754	$12\pi/40$	·81379	$17\pi/40$	·90202
$3\pi/40$	·36959	$8\pi/40$	·67442	$13\pi/40$	·83839	$18\pi/40$	·90978
$4\pi/40$	·44740	$9\pi/40$	·71602	$14\pi/40$	·85938	$19\pi/40$	·91442
$5\pi/40$	·51513	$10\pi/40$	·75288	$15\pi/40$	·87690	$20\pi/40$	·91596

We have thus the means of evaluating all those parts of the expressions which give rise to the most slowly convergent of the series employed.

Taking $\gamma = 2/3$, the values found for u_n', w_n' are tabulated below :—

n.	u_n'	w_n'.
0	− ·13933	+ ·03040
1	− ·01331	− ·00634
2	+ ·00192	+ ·00098
3	+ ·00043	+ ·00022
4	− ·00011	− ·00005
5	− ·00002	− ·00001
6	− ·00001	− ·00000
7	− ·00001	− ·00001
8	+ ·00001	+ ·00001
9	+ ·00001	+ ·00000

Using these and the expressions given above for the finite terms, we can find the values of the displacements on the outer surface of the cylinder.

§ 10. *Numerical Values of the Stresses and Displacements.*

The numerical values obtained in this way are tabulated below; I have given the stresses in the form of ratio (stress) /Q, where Q is the uniform tension which would produce a pull equal to that due to the shear S.

Table of Stresses.

$\widehat{rr}/\mathrm{Q}$.

z.	$r = 0$	$r = (\cdot 2)\,a$	$r = (\cdot 4)\,a$	$r = (\cdot 6)\,a$	$r = a$
0	·22990	·21985	·18587	·11786	·00000
$c/10$ . . .	·22600	·21860	·19244	·13540	·00000
$2c/10$ . . .	·20818	·20842	·20227	·18098	·00000
$3c/10$ . . .	·17064	·17487	·18746	·21264	·00000
$4c/10$ . .	·10688	·10931	·12179	·15054	·00000
$5c/10$ . . .	·02697	·02239	·01531	·00700	·00000
$6c/10$ . . .	– ·04561	– ·05754	– ·08614	– ·13427	·00000
$7c/10$ . . .	– ·09035	– ·10187	– ·13595	– ·18782	·00000
$8c/10$ . . .	– ·08893	– ·09947	– ·12221	– ·14045	·00000
$9c/10$ . . .	– ·05430	– ·05936	– ·06827	– ·06636	·00000
c	·00000	·00000	·00000	·00000	·00000

$\widehat{\phi\phi}/\mathrm{Q}$.

z	$r = 0$	$r = (\cdot 2)\,a$	$r = (\cdot 4)\,a$	$r = (\cdot 6)\,a$	$r = a$
0	·22990	·22924	·22568	·21396	·12363
$c/10$ . . .	·22600	·22631	·22618	·22146	·15364
$2c/10$ . . .	·20818	·21209	·21990	·23357	·22157
$3c/10$ . . .	·17064	·17486	·18827	·21854	·42904
$4c/10$ . . .	·10688	·10865	·11808	·14158	·25536
$5c/10$ . . .	·02697	·02412	·02049	·01498	·00162
$6c/10$ . . .	– ·04561	– ·05310	– ·07023	– ·10563	– ·24868
$7c/10$ . . .	– ·09035	– ·09706	– ·11924	– ·16352	– ·41155
$8c/10$ . . .	– ·08893	– ·09653	– ·11404	– ·14443	– ·18289
$9c/10$ . . .	– ·05430	– ·05836	– ·06709	– ·07983	– ·07880
c	·00000	·00000	·00000	·00000	·00000

$\widehat{zz}/\mathrm{Q}$.

z.	$r = 0$.	$r = (\cdot 2)\,a$.	$r = (\cdot 4)\,a$.	$r = (\cdot 6)\,a$.	$r = a$.
0	·68906	·71895	·81048	·96162	1·11724
$c/10$ . . .	·67272	·70006	·78586	·93696	1·16333
$2c/10$ . . .	·63120	·65168	·71983	·85919	1·34405
$3c/10$ . . .	·58195	·59404	·63659	·73710	2·02246
$4c/10$ . . .	·53943	·54503	·56451	·61686	1·36800
$5c/10$ . . .	·50302	·50502	·50829	·50813	·47865
$6c/10$ . . .	·45713	·45662	·44745	·39955	– ·40866
$7c/10$ . . .	·38411	·38062	·35965	·27810	– 1·05552
$8c/10$ . . .	·27795	·27204	·24438	·15199	– ·35747
$9c/10$ . . .	·14532	·14077	·12060	·06040	– ·13448
c	·00000	·00000	·00000	·00000	·00000

$\widehat{rz}/\mathrm{Q}$.

z.	$r = 0$.	$r = (\cdot 2)\,a$.	$r = (\cdot 4)\,a$.	$r = (\cdot 6)\,a$.	$r = a$.
0	·00000	·00000	·00000	·00000	·00000
$c/10$ . . .	·00000	·02148	·05127	·08883	·00000
$2c/10$ . . .	·00000	·03347	·08205	·15547	·00000
$3c/10$ . . .	·00000	·03262	·08123	·16519	·00000
$4c/10$ . . .	·00000	·02571	·06262	·13159	·95493
$5c/10$ . . .	·00000	·02495	·05701	·11897	·95493
$6c/10$ . . .	·00000	·03718	·08085	·14879	·95493
$7c/10$ . . .	·00000	·05812	·12259	·20528	·00000
$8c/10$ . . .	·00000	·07753	·15596	·23109	·00000
$9c/10$ . . .	·00000	·08891	·16969	·21918	·00000
c	·00000	·09231	·17214	·20992	·00000

In the above tables it is to be remembered that $\widehat{rr}$, $\widehat{zz}$, $\widehat{\phi\phi}$ are all even functions of z; $\widehat{rz}$ is an odd function of z. At the points $r = a$, $z = \pm\frac{c}{3}$, $\widehat{zz}$ and $\widehat{\phi\phi}$ are both $= +\infty$, while at the points $r = a$, $z = \pm\frac{2c}{3}$, $\widehat{zz}$ and $\widehat{\phi\phi}$ are both $= -\infty$.

The displacements u and w have been compared with the corresponding total elongation and lateral contraction w_0 and u_0 of the same cylinder under a uniform tension Q over its plane ends.

TABLE of Displacements.

w/w_0.

z.	$r = 0$.	$r = (\cdot 4)\,a$.	$r = (\cdot 4)\,a$.	$r = (\cdot 6)\,a$.	$r = a$
0	·00000	·00000	·00000	·00000	·00000
$c/10$ . . .	·05693	·06005	·06988	·08685	·10972
$2c/10$ . . .	·11132	·11693	·13489	·16752	·22900
$3c/10$ . . .	·16235	·16949	·19259	·23676	·38253
$4c/10$ . . .	·21132	·21915	·24461	·29489	·59238
$5c/10$ . . .	·26013	·26829	·29467	·34706	·67152
$6c/10$ . . .	·30896	·31741	·34425	·39538	·68809
$7c/10$ . . .	·35493	·36360	·39027	·43724	·57421
$8c/10$ . . .	·39299	·40162	·42700	·46684	·51756
$9c/10$ . . .	·41809	·42646	·45002	·48254	·49745
c	·42684	·43506	·45774	·48725	·49196

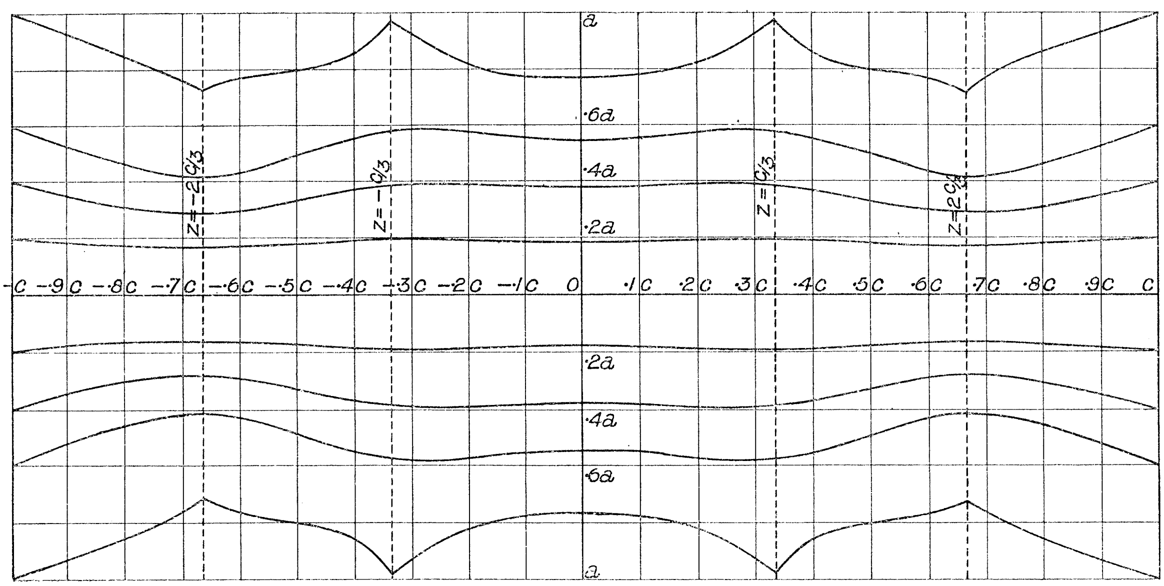
u/u_0.

z.	$r = 0$.	$r = (\cdot 2)a$.	$r = (\cdot 4)a$.	$r = (\cdot 6)a$.	$r = a$.
0	·0000	·0449	·0375	·1341	·5789
$c/10$ . . .	·0000	·0388	·0294	·1120	·5488
$2c/10$ . . .	·0000	·0262	·0170	·0635	·4578
$3c/10$ . . .	·0000	·0202	·0284	·0454	·1847
$4c/10$ . . .	·0000	·0314	·0856	·1206	·3709
$5c/10$ . . .	·0000	·0575	·1767	·2731	·4963
$6c/10$ . . .	·0000	·0838	·2569	·4127	·5861
$7c/10$ . . .	·0000	·0934	·2803	·4466	·5907
$8c/10$ . . .	·0000	·0790	·2313	·3535	·3741
$9c/10$ . . .	·0000	·0446	·1283	·1880	·1807
c	·0000	·0000	·0000	·0000	·0000

§ 11. *Discussion of the Results.*

The numerical results tabulated above are illustrated by the curves contained in Diagrams 1–6. Diagram 1 shows the radial shift, of course enormously exaggerated, u_0 on the diagram being taken as numerically equal to 2/5ths of the radius of the cylinder. For convenience in plotting, the horizontal and vertical scales are not the same, thus $a/5$ and $c/10$ are represented by the same length on the diagram, although their actual ratio is $4/\pi$. The same arrangement has been adhered to in Diagram 2.

Diagram 1.—Distortion of a Cylinder extended by Shearing Stress applied to the Curved Surface (Radial Shifts).

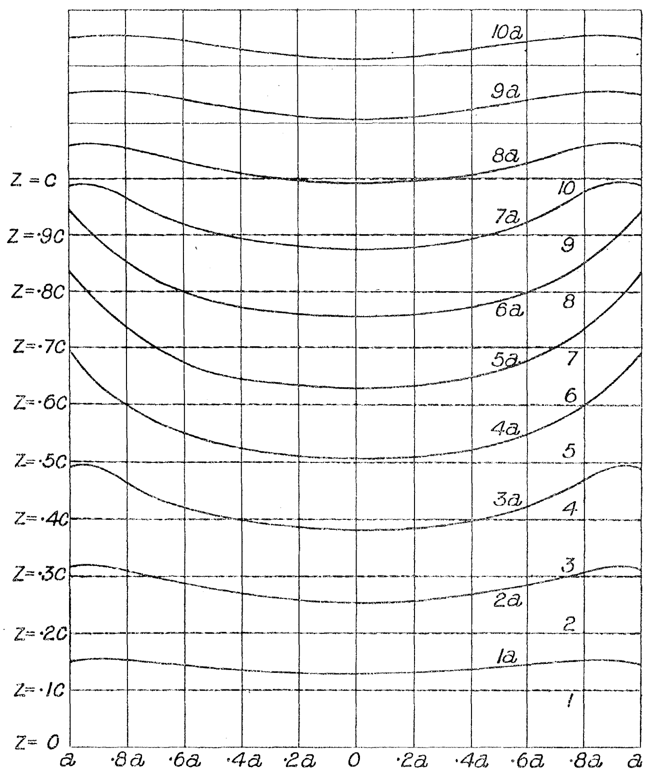


($u_0 = 2a/5$ on the scale of the diagram.)

From Diagram 1 we see at once that a discontinuous change in the slope of the deformed outer surface of the cylinder occurs at the points $z = \pm c/3, \pm 2c/3$, between which the uniform shearing stress is applied. Referring to equation (53) we see that at those points du/dz changes abruptly by the value $-\frac{1}{4}S/\mu$, where S is the abrupt increase in the shear. This result is exhibited in the curves referred to, and we notice that the effect of shear, applied to the outer surface of a cylinder,

Diagram 2.—Distortion of the Cross-sections of a Cylinder under Shearing Stress applied to the Curved Surface.

($w_0 = c/2$ on the scale of the diagram.)



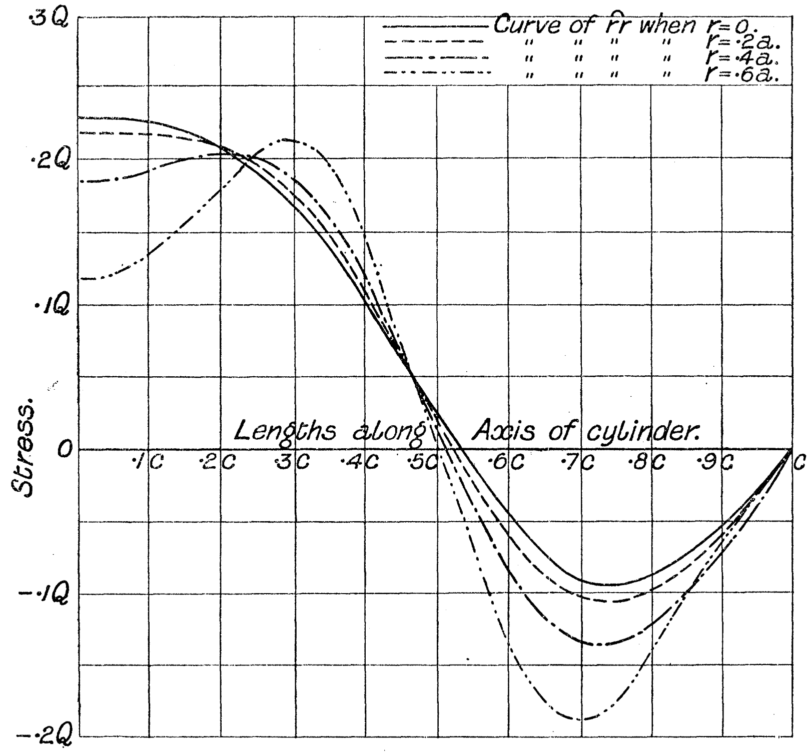
1, 2, 3, 4, 5, 6, 7, 8, 9, 10 undistorted cross-sections.
1a, 2a, 3a, 4a, 5a, 6a, 7a, 8a, 9a, 10a distorted cross-sections.

is to depress that part of the surface towards which the shear is acting. In fact the greatest contraction throughout the cylinder occurs near the points $z = \pm 2c/3$ and appears due to this effect.

Near the ends the cylinder broadens out again, as we should expect, though it is to be noted that the distorted generators meet the plane ends obliquely, which should not be the case if the condition of no stress over the plane ends were accurately fulfilled. This we know is not so: there is a system of finite shear over the plane ends, as is easily seen on referring to the table of $\widehat{rz}$ on p. 172. This system of shears is, however, self-equilibrating. The shear is zero at the centre and at the circumference, and its greatest value does not exceed about 1/4 of the laterally applied shear. Its effects, at some distance inside the cylinder, will therefore be small compared with the effects of the large and unbalanced lateral distribution of shear.

We notice that, for so short a bar, the lateral contraction is very much less than the contraction we should expect according to the "uniform tension" theory. In fact it never amounts to 60 per cent. of that contraction. For points deeper in the material, the contraction is much smaller than this. Thus, for $r = (\cdot 2)\, a$, the lateral contraction is 22 per cent. and for $r = (\cdot 4)\, a$ it is 9 per cent. of what it should be on the "uniform tension" hypothesis. This seems due to the fact, in itself extremely remarkable, that there are considerable radial and cross-radial tensions inside the material. Indeed, referring to Diagram 3, we see that the radial tension amounts to

Diagram 3.—Showing Stress $\widehat{rr}$ for the Cylinder under a Shearing Pull.



about 1/5th of the mean tension Q which would give the same total pull, and which has been consistently taken as the unit of comparison. These tensions are changed to pressures after passing the ring of shear, which is in accordance with the general compressive effect mentioned above.

It may be noticed that the shape of the successive curves on Diagram 3 suggests that, as we approach the outer skin, the two bumps on either side of $z = \cdot 5c$ would lead to infinities, or at all events, to discontinuities in the stress. In other words, that, though we have chosen our constants so as to make $(\widehat{rr})_{r=a}$ formally zero, yet the limit of $(\widehat{rr})$ as given by the series is not zero when r approaches a. This would suggest that the series for $\widehat{rr}$, considered as a series of I-functions, behaves at $r = a$ in much the same way as a discontinuous Fourier series whose general term is $\sin nz$ behaves at $z = \pi$. In fact, if we differentiate $\widehat{rr}$ in the usual way with regard to r and then put $r = a$, we get a divergent series.

It is easily seen, however, that no discontinuity really occurs except at the points where the shear is applied discontinuously. The general term in $\widehat{rr}$ is of the form (dropping irrelevant factors):

$$\frac{\cos \widehat{2n+1}u}{(2n+1)}\left\{\frac{\gamma\alpha[\alpha I_1(\alpha)I_0(\rho) - \rho I_0(\alpha)I_1(\rho)] + (1-\gamma)[I_1(\alpha)I_0(\rho) - \frac{\alpha}{\rho}I_0(\alpha)I_1(\rho)] - \frac{I_1(\rho)}{\rho}[\gamma\alpha^2 I_1(\alpha) - \gamma\rho^2 I_1(\rho)]}{\gamma\alpha^2 I_0^2(\alpha) - (1+\gamma\alpha^2)I_1^2(\alpha)}\right\},$$

where $u = \dfrac{\pi}{2c}(z \pm b \pm e)$.

Now, looking at the semi-convergent expansions for I_0 and I_1 we find, putting $a - d = r$ and $\rho = \alpha - \delta$ where $\delta = \dfrac{\widehat{2n+1}\pi d}{2c}$ and d is small,

$$\frac{I_0(\alpha-\delta)}{I_0(\alpha)} = \epsilon^{-\delta}\left(1 + \tfrac{1}{2}\frac{\delta}{\alpha}\right)\left[1 + \tfrac{1}{8}\frac{\delta}{\alpha^2} + \text{terms of order } \delta/\alpha^3 \text{ and higher terms in } \delta/\alpha\right],$$

and $$\frac{I_1(\alpha-\delta)}{I_1(\alpha)} = \epsilon^{-\delta}\left(1 + \tfrac{1}{2}\frac{\delta}{\alpha}\right)\left[1 - \tfrac{3}{8}\frac{\delta}{\alpha^2} + \text{higher terms}\right],$$

where ϵ = base of Napierian system of logarithms.

The coefficient of $\cos(2n+1)u$ then reduces to the form

$$\frac{1}{(2n+1)}\Bigg[\gamma\alpha\left\{\begin{array}{l} + I_0(\alpha)I_1(\alpha)\alpha \times \dfrac{\delta}{2\alpha^2} \times \epsilon^{-\delta}\left(1 + \text{terms in } \dfrac{\delta}{\alpha} + \text{terms in } \dfrac{\delta}{\alpha^2} + \&\text{c.} + \text{terms in } \dfrac{1}{\alpha} + \ldots\right) \\ + I_0(\alpha)I_1(\alpha)\delta\epsilon^{-\delta}\left(1 + \tfrac{1}{2}\dfrac{\delta}{\alpha} + \text{terms in } \dfrac{\delta}{\alpha^2} \text{ and } \dfrac{\delta}{\alpha^3} + \text{higher terms}\right)\end{array}\right\}$$
$$+ (1-\gamma)I_1(\alpha)I_0(\alpha)\left\{\epsilon^{-\delta}\delta\left(\text{terms of order } \frac{1}{\alpha^2}\right) - \frac{\delta}{\alpha}\epsilon^{-\delta}(\text{a finite term})\right\}$$
$$- \frac{I_1(\rho)}{\rho}\gamma\alpha^2 I_1(\alpha)\left\{\delta\epsilon^{-\delta}\left(\text{terms of order } \frac{1}{\alpha^2}\right) + \frac{2\delta}{\alpha}\epsilon^{-\delta}(\text{a finite term})\right\}\Bigg]$$
$$\div\ (\gamma\alpha^2 I_0^2(\alpha) - (1+\gamma\alpha^2)I_1^2(\alpha)).$$

Now, if we remember that $\gamma\alpha^2 I_0^2(\alpha) - (1 + \gamma\alpha^2) I_1^2(\alpha)$ is of order $\gamma\alpha I_0^2(\alpha)$, we see that the successive terms in the coefficient of $\cos \widehat{2n+1}u$ are of the orders

$$\frac{\delta\epsilon^{-\delta}}{(2n+1)^2}, \quad \frac{\delta\epsilon^{-\delta}}{(2n+1)}, \quad \frac{\delta\epsilon^{-\delta}}{(2n+1)^4}, \quad \frac{\delta\epsilon^{-\delta}}{(2n+1)^3}, \quad \frac{\delta\epsilon^{-\delta}}{(2n+1)^3}, \quad \frac{\delta\epsilon^{-\delta}}{(2n+1)^2},$$

respectively. Also in considering discontinuities, we need only consider the terms towards infinity, for the terms at the beginning can introduce no discontinuity.

But clearly the series

$$\Sigma \frac{\delta\epsilon^{-\delta}}{(2n+1)^3} \cos \widehat{2n+1}\, u, \qquad \Sigma \frac{\delta\epsilon^{-\delta}}{(2n+1)^4} \cos \widehat{2n+1}\, u,$$

are of the order d multiplied by a series, which is finite and continuous up to and including the value $d = 0$. They tend therefore to the limit 0 with d, and can introduce no discontinuity in the stress.

The same will be seen to hold of the series

$$\Sigma \frac{\delta\epsilon^{-\delta}}{(2n+1)^2} \cos \widehat{2n+1}\, u, \qquad \text{provided } u \neq 0.$$

If however $u = 0$, we have to deal with the series

$$\frac{\pi d}{2c} \Sigma \frac{1}{(2n+1)} \epsilon^{-(2n+1)\frac{\pi d}{2c}}.$$

The series under the sign of summation is divergent if $d = 0$. If, however, d is small, but still finite, the series can be summed, and we have the expression equal to

$$\frac{\pi d}{4c} \log \left(\frac{1 + \epsilon^{-\frac{\pi d}{2c}}}{1 - \epsilon^{-\frac{\pi d}{2c}}} \right).$$

This tends to zero when d is small, provided

$$\frac{\pi d}{4c} \log \left(1 - \epsilon^{-\frac{\pi d}{2c}}\right), \quad i.e., \quad \frac{\pi d}{4c} \log \frac{\pi d}{2c} \qquad \text{tend to zero,}$$

which is known to be the case. Hence this series again can never introduce a discontinuity in the stress.

Now consider the series

$$\Sigma \frac{\delta\epsilon^{-\delta}}{2n+1} \cos \widehat{2n+1}u = \frac{\pi d}{2c} \Sigma \epsilon^{-\delta} \cos \widehat{2n+1}u.$$

This is not of the same form. The series under the Σ is sometimes oscillatory, and sometimes divergent, but is never convergent, if d is put equal to zero.

But if d be small but still finite, the series

$$\Sigma\epsilon^{-\frac{\widehat{2n+1}\pi d}{2c}}\cos\widehat{2n+1}u = \frac{(\epsilon^{\frac{\pi d}{2c}} - \epsilon^{\frac{3\pi d}{2c}})\cos u}{1+\epsilon^{4\left(\frac{\pi d}{2c}\right)} - 2\epsilon^{2\left(\frac{\pi d}{2c}\right)}\cos 2u}.$$

As d approaches the limit 0, this series also approaches the limit 0. Hence, *à fortiori*, this series multiplied by d approaches the limit 0, and the stress is continuous.

This holds provided $u \pm 0$. But if $u = 0$ the series in question $= \frac{e^{\pi d/2c}}{1 - e^{\pi d/2c}}$. The limit of $\frac{\pi d}{2c}\frac{e^{\pi d/2c}}{1 - e^{\pi d/2c}}$ when $d = 0$ is $-\frac{1}{2}$. But when $d =$ absolute zero in the series $d\Sigma e^{-\delta}\cos\widehat{2n+1}u$ the series $= 0$ identically.

We have therefore in these cases a finite discontinuity in the stress. This takes place at the points $u = 0$, *i.e.*, $z = \pm b \pm e$, where the shear $\widehat{rz}$ varies discontinuously. At all other points $\widehat{rr}$ approaches the value zero continuously as we move up to the outer surface of the cylinder.

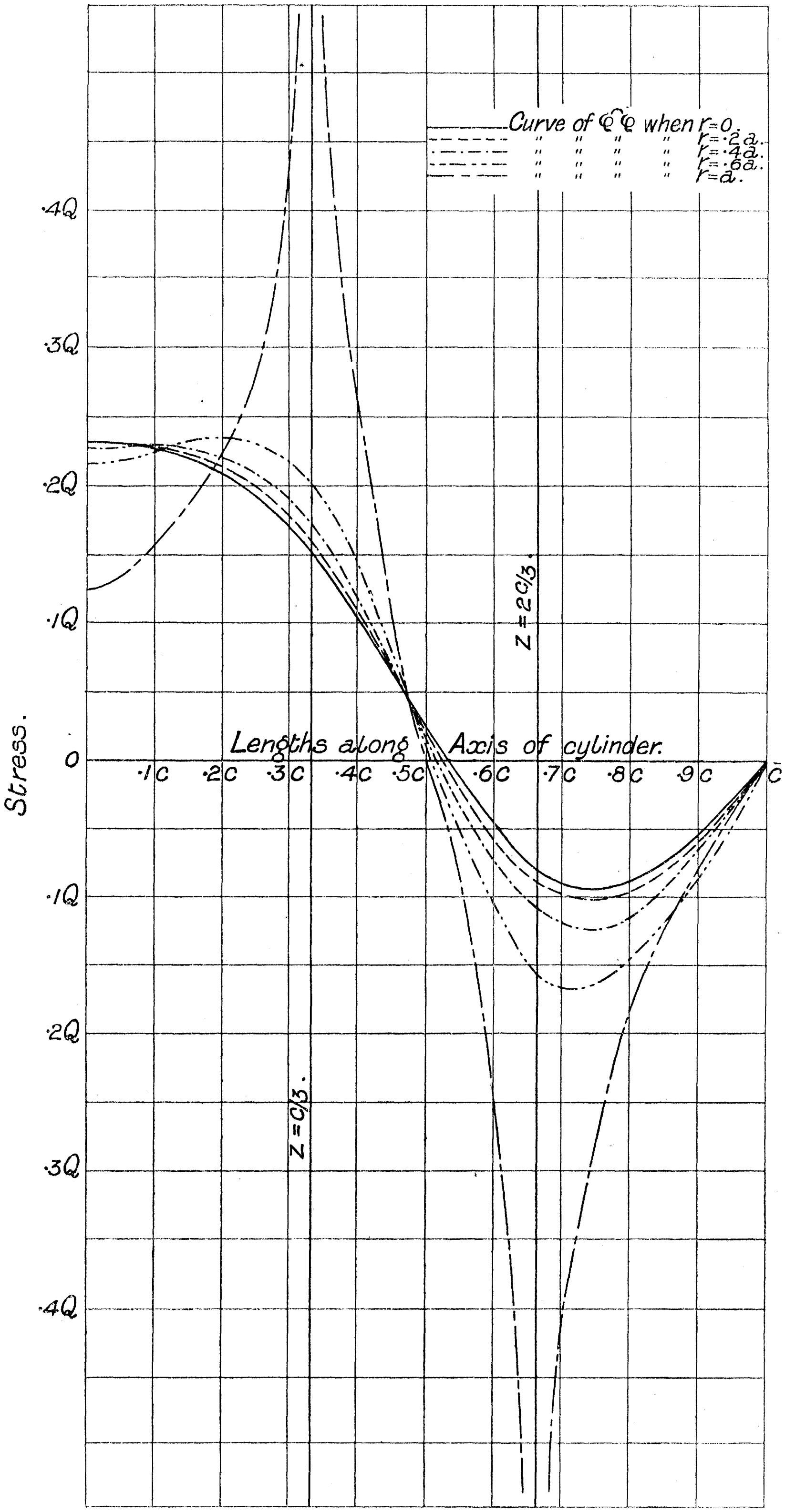
Coming now to the distortion of the cross-sections, this is exhibited in Diagram 2. The displacements are exaggerated, as in Diagram 1, w_0 being taken $= \frac{1}{2}c$. The cross-sections become hollowed out in the middle, the greatest longitudinal extension taking place at the sides. Another noticeable feature is that the cross-sections are slightly curled round the rim, except over the part of the cylinder which is subjected to shear, where they slope up sharply. This follows from the fact that $\frac{dw}{dr} + \frac{du}{dz} = \frac{\widehat{rz}}{\mu}$.

Thus, where $\widehat{rz} = 0$ and $du/dz > 0$ from Diagram 1, it follows that $dw/dr < 0$ or, since $dw/dr > 0$ nearer the centre, a maximum value of w occurs at a comparatively small distance inside the "outer skin" of the cylinder. When, however, $\widehat{rz}$ increases by S, we have seen that du/dz increases by $-\frac{1}{4}\,\mathrm{S}/\mu$, hence dw/dr increases by $\frac{5}{4}\mathrm{S}/\mu$, and is always positive at the outer surface. At the further end, where S ceases to act, the reverse takes place.

It is now easy to understand why the tension is infiuite at the inner end of the shear ring and the pressure infinite at the outer. For if we take two parallel near cross-sections, the one just inside the shear ring and the other just outside, the distorted cross-sections remain sensibly parallel until we approach the outer surface, when they diverge sharply, if near the inside boundary, and converge sharply if at the outside boundary. In the one case we get an infinite extension, in the other an infinite compression. Hence we should expect the stresses $\widehat{zz}$ and $\widehat{\phi\phi}$ to become infinite at these points, and the stress $\widehat{rr}$ to vary infinitely rapidly—and this, we have seen, is what does actually occur.

Further, we see that if we measure the elongation of the outer skin as is done with an extensometer, we shall always get too high a value for the extension. Referring to the table on p. 172, we have the following table of the displacements

Diagram 4.—Showing Stress $\widehat{\phi\phi}$ for the Cylinder under Shearing Pull.



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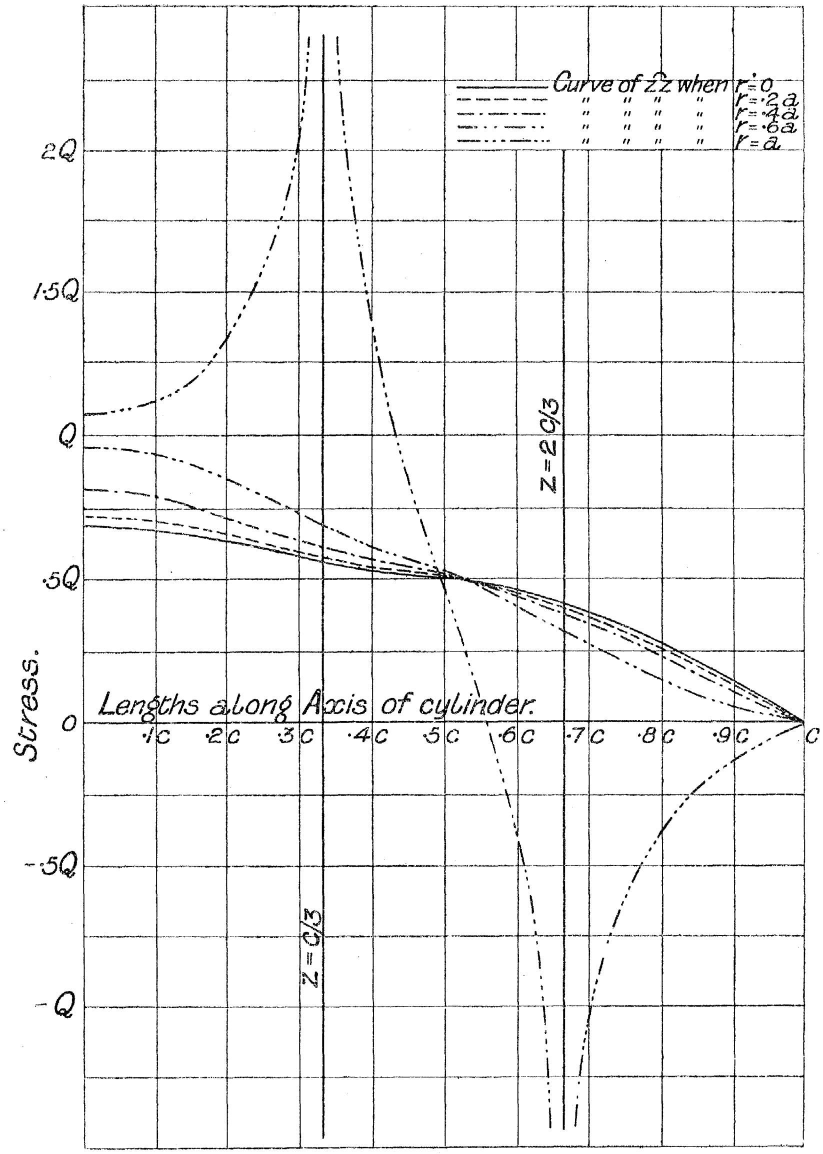
measured by the extensometer, as compared with the displacements calculated from the ordinary theory, over the free length of the bar :—

Displacements.	$z = (\cdot 1) c.$	$z = (\cdot 2) c.$	$z = (\cdot 3) c.$	$z = (\cdot 4) c.$	$z = (\cdot 5) c.$
Actual	·10972	·22900	·38253	·59238	·67152
Calculated	·10000	·20000	·30000	·40000	·50000
Difference	·00972	·02900	·08253	·19238	·17152
Percentage correction	−8·86	−12·66	−21·57	−32·48	−25·54

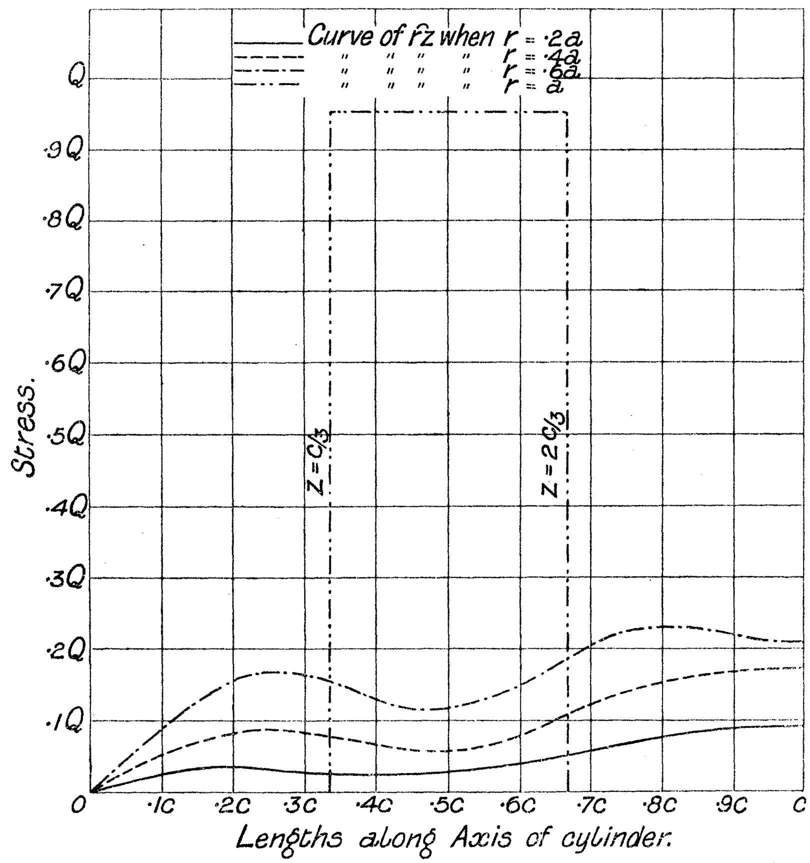
We see, therefore, that in such a case very large corrections have to be applied to extensometer readings.

Diagram 5.—Showing Stress $\widehat{zz}$ for the Cylinder under Shearing Pull.



Diagrams 3–6 give curves showing the variations of the stresses, with z, for the values of r equal to 0, (·2) a, (·4) a, (·6) a, a. I have omitted the intermediate value (·8) a, because the series used converge in this case inconveniently slowly, and no methods of approximation, such as were employed in the case $r = a$, are here available. Observation of the curves for the smaller values of r will, however, in most cases suggest the process by which they are deformed continuously into the curve

Diagram 6.—Showing Stress $\widehat{rz}$ for the Cylinder under Shearing Pull.



corresponding to $r = a$. In Diagram 3, of course, this is not obvious, but here, as has been shown, discontinuous changes occur. In Diagram 6 it is also not quite clear how the curve for $r = (·6)\,a$ becomes transformed into the rectangle corresponding to $r = a$. The curve for $r = (·6)\,a$ has, however, already developed a double hump, and its righthandmost ordinate's rate of increase is fast diminishing. This suggests that the two humps will rise and approach each other, ultimately covering the rectangle, whilst the two "tails" will dwindle down to zero.

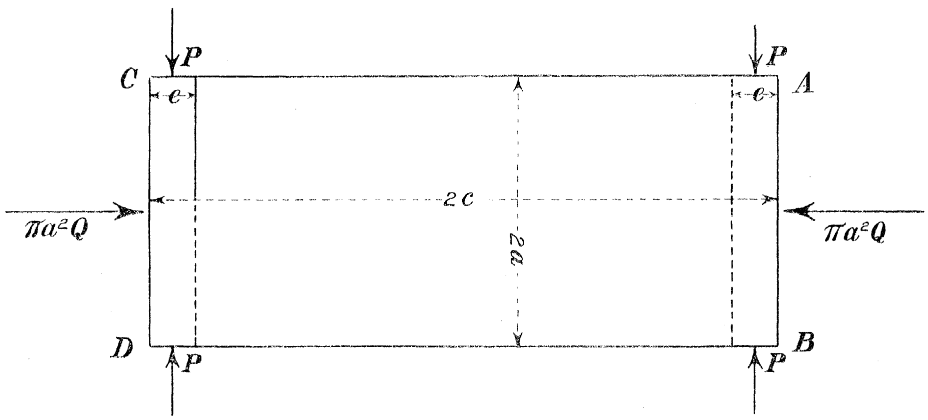
Remarks of a similar character apply to Diagram 5.

§ 12. *The Second Problem: Case of a Cylinder under Pressure whose Ends are not allowed to expand. (First Method of Constraint.)*

Consider a cylinder (fig. 3) subjected to the following system of load :—

(1.) There is no shear $\widehat{rz}$ along the curved surface $r = a$. Over two rings of breadth e at the ends a radial pressure P is made to act, this pressure being so adjusted that there is no radial shift at the points A, B, C, D; the breadth e being in the limit to be made indefinitely small.

Fig. 3



(2.) The plane ends AB, CD, are constrained to remain plane, and are subject to a total normal pressure $\pi a^2 Q$.

The above would fit the case of a cylinder compressed between two rigid planes, into which shallow circular depressions had been cut, to fit the ends of the compressed cylinder and prevent them from expanding.

If we return to the expressions for the stresses in the general case, (24), (25), (26), and also to those for the displacements, we find that if w is to be constant for $z = c$

$$kc = n\pi. \quad \ldots \quad (55),$$

$$C_1 = C_2 = C$$

$$E_1 = E_2 = E = 0$$

$$(A_1 - A_2)\gamma k + 2C = 0 \quad \ldots \quad (56).$$

Also this gives $\widehat{rz} = 0$ over the plane ends, so that we may suppose our rigid constraining plane to be also smooth.

The condition that there is to be no shear over the curved surface now gives

$$(A_1 + A_2) + \frac{2C}{k}\frac{\alpha I_0(\alpha)}{I_1(\alpha)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),$$

(writing $\alpha = ka$, $\rho = kr$ as before).

From (56) and (57)

$$\left.\begin{aligned} A_1 &= -\frac{C}{k}\left(\frac{\alpha I_0}{I_1} + \frac{1}{\gamma}\right) \\ A_2 &= -\frac{C}{k}\left(\frac{\alpha I_0}{I_1} - \frac{1}{\gamma}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

In what follows, the argument of the I-functions, when not written, will always be assumed to be α.

We find

$$\widehat{rr} = 2(\lambda + \mu) u_0 + \lambda w_0 + \Sigma \frac{2\mu C}{k}\left[\begin{array}{l}\left(1 + \frac{\alpha I_0}{I_1}\right) I_0(\rho) \\ -\left\{\left(\frac{\alpha I_0}{I_1} + \frac{1}{\gamma}\right)\frac{I_1(\rho)}{\rho} + \rho I_1(\rho)\right\}\end{array}\right]\cos kz \quad . \quad (59).$$

Putting in this $\rho = \alpha$

$$(\widehat{rr})_{r=a} = 2(\lambda + \mu) u_0 + \lambda w_0 + \Sigma \frac{2\mu C}{k}\frac{\gamma\alpha^2 I_0^2 - (1 + \gamma\alpha^2) I_1^2}{\gamma\alpha I_1}\cos kz \quad . \quad . \quad (60).$$

Now expand the given pressure in the form

$$(\widehat{rr})_{r=a} = P\left(a_0 + \sum_1^\infty a_n \cos\frac{n\pi z}{c}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

Where $a_0, a_1, \ldots a_n, \ldots$ are determined, P remains a free constant.

We have at once, comparing coefficients,

$$\frac{2\mu C}{k} = \frac{\gamma\alpha I_1}{\gamma\alpha^2 I_0^2 - (1 + \gamma\alpha^2) I_1^2} P a_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$2(\lambda + \mu) u_0 + \lambda w_0 = P a_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63).$$

Next we have $u = 0$ when $r = a$, $z = c$

$$0 = u_0 a - \Sigma(-1)^n\left(-\frac{C}{k^2}\left\{\frac{\alpha I_0}{I_1} + \frac{1}{\gamma}\right\} I_1 + \frac{C}{k^2}\alpha I_0\right)$$

$$u_0 a = -P\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64),$$

where $$\zeta = \sum_1^\infty \frac{a_n}{2\mu k}\frac{\alpha I_1^2(-1)^n}{\gamma\alpha^2 I_0^2 - (1 + \gamma\alpha^2) I_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65).$$

This condition gives u_0 in terms of P, and hence by (63)

$$\lambda w_0 = \mathrm{P}\left(a_0 + 2(\lambda + \mu)\frac{\zeta}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66).$$

We have now to find such a value for P that the mean pressure on the plane ends is Q.

$$-\pi a^2 \mathrm{Q} = 2\pi \int_0^a r\widehat{zz}\,dr$$

$$= \pi a^2 (2\lambda u_0 + (\lambda + 2\mu) w_0)$$

$$+ 2\pi \sum_1^\infty \cos\frac{n\pi z}{c}\left\{\left(\{\lambda + 2\mu\}\mathrm{A}_2 - \lambda \mathrm{A}_1 - \lambda\frac{2\mathrm{C}}{k}\right)\int_0^a r\mathrm{I}_0\,dr + 2\mu\mathrm{C}\int_0^a r^2 \mathrm{I}_1\,dr\right\},$$

whence, applying the well-known theorem,

$$\frac{d}{dx}(x^n \mathrm{I}_n(x)) = x^n \mathrm{I}_{n-1}(x),$$

we have

$$-\pi a^2 \mathrm{Q} = \pi a^2 (2\lambda u_0 + (\lambda + 2\mu) w_0)$$

$$+ 2\pi \sum_1^\infty \cos\frac{n\pi z}{c}\left\{\left[(\lambda + 2\mu)\mathrm{A}_2 - \lambda\mathrm{A}_1 - \lambda\frac{2\mathrm{C}}{k}\right]\frac{\alpha \mathrm{I}_1}{k} + 2\mu\mathrm{C}\frac{\alpha^2 \mathrm{I}_2}{k}\right\}.$$

Using the relation

$$\mathrm{I}_2 + \frac{2}{\alpha}\mathrm{I}_1 - \mathrm{I}_0 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67),$$

and putting in for A_1, A_2 their values in terms of C, we find that the terms under the Σ vanish identically. Hence

$$\mathrm{Q} = -(2\lambda u_0 + (\lambda + 2\mu) w_0)$$

$$= \mathrm{P}\left(-\frac{\lambda + 2\mu}{\lambda}a_0 - \frac{2\zeta}{a}(3\lambda + 2\mu)\frac{\mu}{\lambda}\right) \quad . \quad . \quad . \quad . \quad (68).$$

Now suppose the distribution of stress is such that $\widehat{rr} = 0$ from $z = -(c - e)$ to $z = +(c - e)$ and $\widehat{rr} = -\mathrm{P}$ from $z = -c$ to $z = -(c - e)$, and from $z = c - e$ to $z = c$, we find

$$a_0 = -e/c \quad a_n = -\frac{2(-1)^n}{n\pi}\sin\frac{n\pi e}{c},$$

whence

$$\zeta = -\sum_1^\infty \frac{c}{\mu}\frac{\sin\frac{n\pi e}{c}}{n^2\pi^2}\frac{\alpha \mathrm{I}_1{}^2}{\gamma\alpha^2 \mathrm{I}_0{}^2 - (1 + \gamma\alpha^2)\mathrm{I}_1{}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69).$$

When α is at all large, the terms of ζ are comparable with those of the series

$$-\frac{c}{\mu\gamma}\sum_1^\infty \frac{\sin\frac{n\pi e}{c}}{n^2\pi^2},$$

which is equal to

$$-\frac{c}{\pi^2\mu\gamma}\left\{-\frac{\pi e}{c}\log\left(2\sin\frac{\pi e}{2c}\right)+\int_0^{\pi e/c}\frac{x}{2}\cot\frac{x}{2}\,dx\right\} \quad . \quad . \quad . \quad . \quad . \quad (70).$$

(70) will give the approximate value of ζ whenever $\pi a/c$ is at all large. If $a = 2c$, this method of approximation will already be quite fair.

We see therefore that if e tends to zero, ζ also tends to zero, but ζ/e tends to become logarithmically infinite.

Now from (68)

$$\mathrm{P}a_0 = \frac{-\lambda \mathrm{Q} a_0}{(\lambda+2\mu)a_0 + \frac{2\mu\zeta}{a}(3\lambda+2\mu)}$$

$$= \frac{-\lambda\mathrm{Q}}{(\lambda+2\mu) - \frac{2\mu c}{a}\left(\frac{\zeta}{e}\right)(3\lambda+2\mu)}.$$

Hence, since $-\zeta/e$ tends to ∞ when e tends to zero, $\mathrm{P}a_0$ tends to zero when e tends to zero.

And similarly, for any finite value of n, $\mathrm{P}a_n$ tends to zero when e tends to zero.

But if we write down the expressions for the stresses, they are:

$$\left.\begin{aligned}
\widehat{zz} &= -\mathrm{Q} + \Sigma\frac{\mathrm{P}a_n\gamma\alpha\mathrm{I}_1(\alpha)}{\gamma\alpha^2\mathrm{I}_0^2-(1+\gamma\alpha^2)\mathrm{I}_1^2}\left[\rho\mathrm{I}_1(\rho)+\mathrm{I}_0(\rho)\left(2-\frac{\alpha\mathrm{I}_0}{\mathrm{I}_1}\right)\right]\cos\frac{n\pi z}{c}\\
\widehat{rr} &= \mathrm{P}a_0 + \Sigma\frac{\mathrm{P}a_n\gamma\alpha\mathrm{I}_1}{\gamma\alpha^2\mathrm{I}_0^2-(1+\gamma\alpha^2)\mathrm{I}_1^2}\left[\left(\frac{\alpha\mathrm{I}_0}{\mathrm{I}_1}+1\right)\mathrm{I}_0(\rho)-\mathrm{I}_1(\rho)\left\{\frac{\alpha\mathrm{I}_0}{\rho\mathrm{I}_1}+\frac{1}{\gamma\rho}+\rho\right\}\right]\cos\frac{n\pi z}{c}\\
\widehat{\phi\phi} &= \mathrm{P}a_0 + \Sigma\frac{\mathrm{P}a_n\gamma\alpha\mathrm{I}_1}{\gamma\alpha^2\mathrm{I}_0^2-(1+\gamma\alpha^2)\mathrm{I}_1^2}\left[\left(\frac{\alpha\mathrm{I}_0}{\mathrm{I}_1}+\frac{1}{\gamma}\right)\frac{\mathrm{I}_1(\rho)}{\rho}-\left(\frac{1}{\gamma}-1\right)\mathrm{I}_0(\rho)\right]\cos\frac{n\pi z}{c}\\
\widehat{rz} &= \Sigma\frac{\mathrm{P}a_n\gamma\alpha\mathrm{I}_1}{\gamma\alpha^2\mathrm{I}_0^2-(1+\gamma\alpha^2)\mathrm{I}_1^2}\left[\rho\mathrm{I}_0(\rho)-\frac{\alpha\mathrm{I}_0}{\mathrm{I}_1}\mathrm{I}_1(\rho)\right]\sin\frac{n\pi z}{c}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (71).$$

Now the above series are absolutely convergent for all values of r except $r = a$, where indeed they are discontinuous. Leaving the neighbourhood of $r = a$ out of account, we see that for points inside the material, when the space over which the constraining pressure acts is indefinitely reduced, $\mathrm{LP}a_n = 0$ and

$$\widehat{zz} = -\mathrm{Q}$$
$$\widehat{rz} = \widehat{rr} = \widehat{\phi\phi} = 0;$$

therefore outside the rim, where plastic deformation may be expected to occur, the stresses are exactly the same as on the ordinary hypothesis.

We come then to the conclusion that this method of preventing the ends from expanding is not adequate, and that to obtain any real effect, we require to make the constraining rim of a certain definite thickness.

In so doing, we are really introducing an additional condition, besides the non-expansion of the ends, the cylinder being now, as it were, built-in. The problem as it stands did not appear of sufficient interest to warrant the expenditure of arithmetical labour upon it, so I have contented myself with stating the algebraical results.

§ 13. *The Second Problem: Constraint effected by Shear over the Terminal Cross-sections. Determination of the Constants.*

Suppose now that we consider our cylinder subject to the following conditions :—

(i.) A total pressure $\pi a^2 Q$ over the plane ends, the distribution of this pressure being unknown.

(ii.) The ends constrained to remain plane, so that $w =$ const. when $z = \pm c$.

(iii.) The ends not to expand along the perimeter

$$u = 0 \text{ when } r = a,\ z = \pm c.$$

This condition is satisfied by allowing a shear $\widehat{rz}$ over the plane ends, its distribution being, however, unknown.

(iv.) No stress across the curved surface, *i.e.*,

$$\widehat{rr} = 0 \text{ when } r = a,$$

$$\widehat{rz} = 0 \text{ when } r = a.$$

These conditions will represent the state of things which we may expect to hold it the cylinder be compressed between two rigid planes which are sufficiently rough to prevent the expansion of the ends.

Now, in such a case as this, it is obvious that the expressions for the stresses and strains as purely periodic series in z break down, for if we take the expressions (24) and (26) for $\widehat{rr}$ and $\widehat{rz}$ the condition that $w =$ const. when $z = \pm c$ will give us, as before, $E = 0$, and the vanishing of the stresses at the curved surface will give two homogeneous equations of condition between A_1, A_2, and C. These, taken in conjunction with equation (35), give three linear homogeneous equations in A_1, A_2, and C, which are in general inconsistent unless $A_1 = 0$, $A_2 = 0$, $C = 0$, which would destroy the periodic solution altogether.

We have therefore to assume that u and w are made up of two parts. The first part, which I shall denote by U, W, consists of the periodic solution hitherto obtained. The second part is a finite power series in r and z. The resulting expression is a combination of the two types of solution, which are discussed separately by Mr. CHREE ('Camb. Phil. Soc. Trans.,' vol. 14). Either of these two types, taken by itself, is of comparatively restricted application, but by combining the two we are enabled to deal with far more general problems.

Assume therefore

$$u = u_0 r + \frac{u_1 r^3}{3} + \frac{u_2 r^5}{5} + \frac{\mathrm{D} r z^2}{2} + \frac{\mathrm{E} r^3 z^2}{2} + \frac{\mathrm{F} r z^4}{4} + \mathrm{U} \quad . \quad . \quad . \quad . \quad . \quad . \quad (72),$$

$$w = w_0 z + \frac{w_1 z^3}{3} + \frac{w_2 z^5}{5} + \frac{\mathrm{D}' r^2 z}{2} + \frac{\mathrm{E}' r^2 z^3}{2} + \frac{\mathrm{F}' r^4 z}{4} + \mathrm{W} \quad . \quad . \quad . \quad . \quad . \quad (73).$$

The above power series are the most general expressions of the fifth degree consistent with the conditions that u must be odd in r and even in z, and w must be odd in z and even in r.

In the above we have, as before,

$$\mathrm{U} = \Sigma \left\{ -\frac{\mathrm{A}_1}{k} \mathrm{I}_1(kr) - \frac{\mathrm{C}}{k} r \mathrm{I}_0(kr) \right\} \cos kz \quad . \quad . \quad . \quad . \quad . \quad (74),$$

$$\mathrm{W} = \Sigma \left\{ \frac{\mathrm{A}_2}{k} \mathrm{I}_0(kr) + \frac{\mathrm{C}}{k} r \mathrm{I}_1(kr) \right\} \sin kz \quad . \quad . \quad . \quad . \quad . \quad . \quad (75).$$

Consider first of all the condition that w is to be constant when $z = \pm c$. This fixes k:

$$k = n\pi/c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76).$$

Further we have

$$\mathrm{F}' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77),$$

$$\tfrac{1}{2}\mathrm{D}'c + \tfrac{1}{2}\mathrm{E}'c^3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78).$$

Now remember that u and w have to satisfy the differential equations

$$(\lambda + 2\mu) \frac{d}{dr} \frac{1}{r} \frac{d}{dr}(ru) + \mu \frac{d^2 u}{dz^2} + (\lambda + \mu) \frac{d^2 w}{dr\,dz} = 0,$$

$$(\lambda + \mu) \frac{1}{r} \frac{d}{dr} r \frac{du}{dz} + \mu \frac{1}{r} \frac{d}{dr}\left(r \frac{dw}{dr}\right) + (\lambda + 2\mu) \frac{d^2 w}{dz^2} = 0.$$

The parts U and W we have seen already will satisfy these equations, provided

$$\mathrm{A}_1 - \mathrm{A}_2 + 2\mathrm{C}/\gamma k = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79).$$

Consider therefore only the algebraic terms. Of these $u_0 r$ and $w_0 z$ always satisfy the above equations.

The third order terms require

$$\tfrac{8}{3} u_1 (\lambda + 2\mu) + \mu \mathrm{D} + (\lambda + \mu)\,\mathrm{D}' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80),$$

$$2(\lambda + \mu)\,\mathrm{D} + 2\mu \mathrm{D}' + (\lambda + 2\mu)\, 2w_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (81).$$

The fifth order terms give

$$(\lambda + 2\mu)\left(\frac{24}{5}u_2 r^3 + 4\mathrm{E}rz^2\right) + \mu\mathrm{E}r^3 + 3\mu\mathrm{F}rz^2 + (\lambda + \mu)(3\mathrm{E}'rz^2) = 0,$$

$$(\lambda + \mu)(4\mathrm{E}r^2 z + 2\mathrm{F}z^3) + 2\mu\mathrm{E}'z^3 + (\lambda + 2\mu)[4w_2 z^3 + 3\mathrm{E}'r^2 z] = 0$$

which imply the four relations

$$(\lambda + 2\mu)\frac{24u_2}{5} + \mu\mathrm{E} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (82),$$

$$2(\lambda + \mu)\mathrm{F} + 2\mu\mathrm{E}' + (\lambda + 2\mu)4w_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (83),$$

$$4\mathrm{E}(\lambda + 2\mu) + 3\mu\mathrm{F} + 3\mathrm{E}'(\lambda + \mu) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (84),$$

$$4\mathrm{E}(\lambda + \mu) + 3\mathrm{E}'(\lambda + 2\mu) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (85).$$

There is, however, a further relation to be satisfied among these constants, and that is obtained as follows. If we proceed to write down the expressions for $\widehat{rr}$ and $\widehat{rz}$ and to put in them $r = a$, we shall obtain expressions of the form

$$\widehat{rr} = \text{algebraic polynomial in } z + \text{series of cosines of } n\pi z/c,$$

$$\widehat{rz} = \text{algebraic polynomial in } z + \text{series of sines of } n\pi z/c,$$

where the coefficients of cos $n\pi z/c$, sin $n\pi z/c$, contain the two undetermined constants A_1 and A_2.

We may now proceed to expand the two polynomials in series of cosines or sines of $n\pi z/c$. Equating then the coefficient of each cosine and sine to zero, we can make $\widehat{rr}$ and $\widehat{rz}$ zero over the whole of the curved surface, and at the same time we obtain two equations for A_1 and A_2.

But it is clear that, if the Fourier expressions in the second case are to be continuous, then the algebraic polynomial part of $\widehat{rz}$ must reduce to zero when $z = \pm c$, otherwise its expansion in sines of $n\pi z/c$ is discontinuous, and at the perimeter of the flat ends the shear is discontinuous. This introduces infinite stresses at this point which render the solution inconvenient.

Now we have at our disposal *nine* constants; these have already been made to satisfy the seven homogeneous equations (78), (80)–(85), and therefore we are free to make them satisfy an eighth homogeneous equation.

Choose then the constants so as to make (polynomial part of $[du/dz + dw/dr]$ when $r = a$, $z = \pm c$) zero, and we have

$$\mathrm{D}ac + \mathrm{E}a^3c + \mathrm{F}ac^3 + \mathrm{D}'ac + \mathrm{E}'ac^3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (86).$$

If now we express all the other constants in terms of the constant E, we find ;

$$\left.\begin{aligned} w_1 &= \left[a^2 - \frac{8c^2}{3}\right]\gamma E, & u_1 &= \tfrac{3}{8}\left[(1-\gamma)\,a^2 - \tfrac{4}{3}c^2\right]E, \\ w_2 &= \tfrac{4}{3}\gamma E, & u_2 &= -\tfrac{5}{24}(1-\gamma)\,E, \\ D &= \{\tfrac{4}{3}(1+\gamma)\,c^2 - a^2\}E, & D' &= \frac{4\gamma c^2}{3}E, \\ F &= -\tfrac{4}{3}(1+\gamma)\,E, & E' &= -\tfrac{4}{3}\gamma E \end{aligned}\right\} \quad . \quad . \quad . \quad (87),$$

and these satisfy the equations and the condition (86).

It is noticeable that a solution can be obtained, in the form

$$u = u_0 r + \frac{u_1 r^3}{3} + \frac{Drz^2}{2} + U,$$

$$w = w_0 z + \frac{w_1 z^3}{3} + W,$$

which can be made to satisfy all the conditions except (86). If, however, one works out this solution, it is found, as we should expect, to give infinite values for the stresses, all round the perimeter of the plane ends. Thus, though simpler in form, this solution is not really simpler to work with. I have given on pp. 217–219 the expressions for the stresses and displacements obtained from such a solution.

§ 14. *Determination of the Coefficients so as to Satisfy the Conditions at the Curved Surface.*

If we write down the expressions for the stresses, we find

$$\left(\frac{\widehat{rz}}{\mu}\right)_{r=a} = (D + D')\,az + Ea^3z + (E' + F)\,az^3 + \frac{dU}{dz} + \frac{dW}{dr}.$$

We have therefore to make

$$-(D + D')\,az - Ea^3z - (E' + F)\,az^3$$
$$= \sum_1^\infty \left((A_1 + A_2)\,I_1(\alpha) + \frac{2C}{k}\alpha I_0(\alpha)\right)\sin\frac{n\pi z}{c}.$$

Now we find easily

$$z = \sum_1^\infty (-1)^{n-1}\frac{2c}{n\pi}\sin\frac{n\pi z}{c},$$

$$z^3 = \sum_1^\infty (-1)^{n-1}\left(\frac{2c^3}{n\pi} - \frac{12c^3}{n^3\pi^3}\right)\sin\frac{n\pi z}{c}.$$

Hence if we expand $-(D + D')\,az - Ea^3z - (E' + F)\,az^3$ in a series of the form $\sum_1^\infty a_n \sin\frac{n\pi z}{c}$ we find

$$a_n = [-(D + D')\,ac - Ea^3c - (E' + F)\,ac^3](-1)^{n-1} \times \frac{2}{n\pi}.$$
$$+ (-1)^{n-1}\frac{12ac^3}{n^3\pi^3}(E' + F).$$

The first term is zero in virtue of equation (86), and using the relations (87) we find

$$a_n = (-1)^n \frac{16ac^3}{n^3\pi^3} \times \mathrm{E} \times (2\gamma + 1),$$

whence, comparing coefficients

$$\mathrm{A}_1 + \mathrm{A}_2 + \frac{2\mathrm{C}}{k}\frac{\alpha \mathrm{I}_0}{\mathrm{I}_1} = (-1)^n \frac{16ac^3}{n^3\pi^3}(2\gamma + 1)\,\mathrm{E} \quad . \quad . \quad . \quad . \quad . \quad (88).$$

This gives us $\widehat{rz}$ consistently zero right up to the plane end.

Next we have

$$\begin{aligned}(\widehat{rr})_{r=a} = {} & 2(\lambda + \mu)\,u_0 + \lambda w_0 \\ & + a^2\left[\tfrac{2}{3}(2\lambda + 3\mu)\,u_1 + \frac{\mathrm{D}'}{2}\lambda\right] + z^2[(\lambda + \mu)\,\mathrm{D} + \lambda w_1] \\ & + \frac{a^2z^2}{4}[(4\lambda + 6\mu)\,\mathrm{E} + 3\lambda\mathrm{E}'] + a^4\left(\frac{6\lambda + 10\mu}{5}\right)u_2 \\ & + z^4\left[\frac{\lambda + \mu}{2}\mathrm{F} + \lambda w_2\right] + (\lambda + 2\mu)\left(\frac{d\mathrm{U}}{dr}\right)_{r=a} + \frac{\lambda \mathrm{U}}{a} + \lambda\left(\frac{d\mathrm{W}}{dz}\right)_{r=a}.\end{aligned}$$

Hence we have to make

$$\begin{aligned}& \sum_1^\infty \left\{-\mu[(1+\gamma)\mathrm{A}_1 + (1-\gamma)\mathrm{A}_2]\mathrm{I}_0 + 2\mu\left[\mathrm{A}_1\frac{\mathrm{I}_1}{a} - \frac{\mathrm{C}}{k}\alpha\mathrm{I}_1\right]\right\}\cos\frac{n\pi z}{c} \\ & = -2(\lambda + \mu)u_0 - \lambda w_0 \\ & - a^2\left[\tfrac{2}{3}(2\lambda + 3\mu)\,u_1 + \frac{\mathrm{D}'\lambda}{2}\right] - a^4\left(\frac{6\lambda + 10\mu}{5}\right)u_2 \\ & - z^2\left\{(\lambda + \mu)\mathrm{D} + \lambda w_1 + \frac{a^2}{2}[(4\lambda + 6\mu)\mathrm{E} + 3\lambda\mathrm{E}']\right\} \\ & - z^4\left[\frac{\lambda + \mu}{2}\mathrm{F} + \lambda w_2\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89).\end{aligned}$$

Now we have

$$z^2 = \frac{c^2}{3} + \sum_1^\infty \frac{4c^2(-1)^n}{n^2\pi^2}\cos\frac{n\pi z}{c},$$

$$z^4 = \frac{c^4}{5} + \sum_1^\infty 8c^4(-1)^n\left(\frac{1}{n^2\pi^2} - \frac{6}{n^4\pi^4}\right)\cos\frac{n\pi z}{c}.$$

Hence if the right-hand side of equation (89) be expanded in the form

$$b_0 + \sum_1^\infty b_n \cos\frac{n\pi z}{c},$$

we find

$$b_0 = -2(\lambda+\mu)u_0 - \lambda w_0$$
$$- a^2\left[\tfrac{2}{3}(2\lambda+3\mu)u_1 + \frac{D'}{2}\lambda\right] - a^4\left(\frac{6\lambda+10\mu}{5}\right)u_2$$
$$- \frac{c^2}{3}\left\{(\lambda+\mu)D + \lambda w_1 + \frac{a^2}{2}\left[(4\lambda+6\mu)E + 3\lambda E'\right]\right\}$$
$$- \frac{c^4}{5}\left[\frac{\lambda+\mu}{2}F + \lambda w_2\right]$$

$$b_n = -\frac{(-1)^n 4c^2}{n^2\pi^2}\left\{(\lambda+\mu)D + \lambda w_1 + \frac{a^2}{2}[(4\lambda+6\mu)E + 3\lambda E']\right\}$$
$$- \frac{8c^4(-1)^n}{n^2\pi^2}\left\{\frac{\lambda+\mu}{2}F + \lambda w_2\right\} + \frac{48c^4}{n^4\pi^4}(-1)^n\left\{\frac{\lambda+\mu}{2}F + \lambda w_2\right\},$$

whence using equations (87) we find after long but otherwise straightforward reductions

$$b_0 = -2(\lambda+\mu)u_0 - \lambda w_0$$
$$+ \mu E\left\{-\tfrac{14}{15}c^4\gamma - \frac{a^4}{12}(2\gamma+1) + \frac{2a^2c^2\gamma}{3}\right\},$$

$$b_n = -(-1)^n\frac{4c^2a^2}{n^2\pi^2}\mu E(2\gamma+1) - \frac{96c^4}{n^4\pi^4}(-1)^n E\mu\gamma.$$

Hence, equating coefficients on both sides of equation (89) we obtain the relations

$$2(\lambda+\mu)u_0 + \lambda w_0 = \mu E\left\{-\tfrac{14}{15}c^4\gamma - \frac{a^4}{12}(2\gamma+1) + \frac{2a^2c^2\gamma}{3}\right\} \quad . \quad . \quad (90),$$

$$-[(1+\gamma)A_1 + (1-\gamma)A_2]I_0 + 2\left[A_1\frac{I_1}{\alpha} - \frac{C}{k}\alpha I_1\right]$$
$$= -(-1)^n E \times \frac{4c^2a^2}{n^2\pi^2}\left(2\gamma + 1 + \frac{24c^2\gamma}{a^2n^2\pi^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91),$$

and it in (91) we substitute for A_1, A_2 their values in terms of C/k deduced from (79) and (88), we have

$$\frac{2\mu C}{k}\,\frac{\alpha^2\gamma I_0^2 - I_1^2(1+\alpha^2\gamma)}{\alpha\gamma I_1}$$
$$= -(-1)^n\mu E \times \frac{4c^2a^2}{n^2\pi^2}\left[2\gamma + 1 + \frac{4c^2}{a^2n^2\pi^2}(8\gamma+1) - \frac{4cI_0(2\gamma+1)}{an\pi I_1}\right],$$

or

$$\frac{2\mu C}{k} = -(-1)^n \frac{\mu\gamma E 4a^2}{k^2} \frac{\left\{\left[(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right] I_1 - 4I_0(2\gamma+1)\right\}}{\gamma\alpha^2 I_0^2 - (1+\gamma\alpha^2) I_1^2} \quad . \quad . \quad (92).$$

A_1 and A_2 being known in terms of C, all the constants are now determined, except u_0, w_0, and E.

§ 15. *Determination of the Constants* u_0, w_0, E.

Let us now apply the condition that :

$$\int_0^{2\pi} d\theta \int_0^a r\widehat{zz}dr = -\pi a^2 Q.$$

If we write down the expression for $\widehat{zz}$, using the expressions for A_1 and A_2 in terms of C, we find :

$$\widehat{zz} = 2\lambda u_0 + (\lambda + 2\mu) w_0$$

$$+ \left[\frac{4\lambda}{3} u_1 + (\lambda + 2\mu)\frac{D'}{2}\right] r^2 + [\lambda D + (\lambda + 2\mu) w_1] z^2 + \left[2E\lambda + \frac{3E'}{2}(\lambda + 2\mu)\right] r^2 z^2$$

$$+ \frac{6\lambda}{5} u_2 r^4 + \left[\frac{\lambda F}{2} + (\lambda + 2\mu) w_2\right] z^4$$

$$+ \sum_1^\infty \frac{2\mu C}{k}\left[\rho I_1(\rho) - I_0(\rho)\left\{\frac{\alpha I_0}{I_1} - 2\right\}\right] \cos\frac{n\pi z}{c}$$

$$+ (2\gamma + 1)\mu E \sum_1^\infty \frac{I_0(\rho)}{I_1}(-1)^n \times \frac{16ac^3}{n^3\pi^3} \cos\frac{\pi n z}{c}$$

$$= 2\lambda u_0 + (\lambda + 2\mu) w_0$$

$$+ \mu E \left\{ \begin{array}{l} \left[\frac{2}{3}c^2 + (2\gamma - 1)\frac{a^2}{2}\right] r^2 + \left[a^2 - \frac{4}{3}c^2(2\gamma + 1)\right] z^2 - 2r^2 z^2 \\ \qquad - \frac{1}{12}(2\gamma - 1) r^4 + \frac{2}{3}(2\gamma + 1) z^4 \end{array} \right\}$$

$$+ \sum_1^\infty \frac{2\mu C}{k}\left[\rho I_1(\rho) - I_0(\rho)\left\{\frac{\alpha I_0}{I_1} - 2\right\}\right] \cos\frac{n\pi z}{c}$$

$$+ (2\gamma + 1)\mu E \sum_1^\infty \frac{I_0(\rho)}{I_1}(-1)^n \times \frac{16ac^3}{n^3\pi^3} \cos\frac{n\pi z}{c}.$$

Hence

$$\int_0^{2\pi} d\theta \int_0^a \widehat{zz} r dr = \pi a^2 (2\lambda u_0 + (\lambda + 2\mu) w_0)$$

$$+ \mu E \times 2\pi \left\{ \begin{array}{l} \frac{a^4}{4}\left[\frac{(2\gamma-1)}{2}a^2 + \frac{2}{3}c^2\right] - \frac{z^2 a^2}{2}\left[\frac{4}{3}(2\gamma+1)c^2 - a^2\right] - \frac{a^6}{6}\frac{2\gamma-1}{4} - \frac{a^4 z^2}{2} \\ + a^2 z^4 \frac{(2\gamma+1)}{3} + 16a(2\gamma+1)\sum_1^\infty (-1)^n \frac{c^5}{n\pi^5} \frac{\int_0^a \rho I_0(\rho)\,d\rho}{I_1(\alpha)} \cos\frac{n\pi z}{c} \end{array} \right\}$$

$$+ 2\pi \sum_1^\infty \frac{2\mu C}{k^3} \int_0^a \left[\rho^2 I_1(\rho) - \rho I_0(\rho)\left\{\frac{\alpha I_0}{I_1} - 2\right\}\right] d\rho \cos\frac{n\pi z}{c}.$$

But

$$\int_0^\alpha \left\{\rho^2 I_1(\rho) - \rho I_0(\rho)\left[\frac{\alpha I_0}{I_1} - 2\right]\right\} d\rho = \alpha^2 I_2(\alpha) - \alpha I_1(\alpha)\left\{\frac{\alpha I_0}{I_1} - 2\right\} = 0 \text{ by (67)},$$

and

$$\int_0^\alpha \rho I_0(\rho)\,d\rho = \alpha I_1(\alpha),$$

so that we have to make

$$2\lambda u_0 + (\lambda + 2\mu) w_0$$
$$+ \mu E \left\{ \begin{array}{l} \frac{a^2}{4}\left[\frac{2\gamma-1}{2}a^2 + \frac{2}{3}c^2\right] - \frac{a^4}{6}\frac{2\gamma-1}{4} - \frac{(2\gamma+1)}{3}(2z^2c^2 - z^4) \\ + 16(2\gamma+1)c^4 \sum_1^\infty \frac{(-1)^n}{n^4\pi^4}\cos\frac{n\pi z}{c} \end{array} \right\} = -Q.$$

Now it is easy to show that

$$\sum_1^\infty \frac{(-1)^n}{n^4}\cos\frac{n\pi z}{c} = -\frac{z^4\pi^4}{48c^4} + \frac{z^2\pi^4}{24c^2} - \frac{7\pi^4}{720},$$

whence finally

$$2\lambda u_0 + (\lambda + 2\mu) w_0 + 2\mu E\left\{\frac{a^4(2\gamma-1)}{12} + \frac{1}{6}a^2c^2 - \frac{7}{45}(2\gamma+1)c^4\right\} = -Q \quad . \quad . \ (93).$$

(90) and (93) thus give us already two equations for u_0, w_0, and E. We require a third equation.

This is obtained from the condition that

$$(u)_{\substack{z=c \\ r=a}} = 0.$$

This gives

$$u_0 a + \frac{u_1 a^3}{3} + \frac{u_2 a^5}{5} + \frac{\mathrm{D}ac^2}{2} + \frac{\mathrm{E}a^3c^2}{2} + \frac{\mathrm{F}ac^4}{4}$$

$$+ \Sigma \left\{ - \frac{\mathrm{A}_1}{k} \mathrm{I}_1 - \frac{\mathrm{C}}{k^2} \alpha \mathrm{I}_0 \right\} (-1)^n = 0$$

$$u_0 a + \frac{a^3}{8} [(1-\gamma) a^2 - \tfrac{4}{3} c^2] \mathrm{E} - \frac{a^5}{24} (1-\gamma) \mathrm{E} + \frac{ac^2}{2} [\tfrac{4}{3} (1+\gamma) c^2 - a^2] \mathrm{E}$$

$$+ \frac{\mathrm{E}a^3c^2}{2} - \tfrac{1}{3} (1+\gamma) \mathrm{E}ac^4 - (2\gamma+1) \sum_1^\infty \frac{8ac^4}{n^4\pi^4} \mathrm{E} + \sum_1^\infty \frac{\mathrm{C}}{k^2} \frac{\mathrm{I}_1}{\gamma} (-1)^n = 0.$$

Now

$$\sum_1^\infty (-1)^n \frac{\mathrm{C}\mathrm{I}_1}{k\gamma} = - \frac{a^5}{12} \mathrm{E}\zeta,$$

where

$$\zeta = \sum_{n=1}^\infty \frac{24}{\alpha^3} \frac{\left[(2\gamma+1)\alpha + \dfrac{4}{\alpha}(8\gamma+1) \right] \mathrm{I}_1^2 - (8\gamma+4) \mathrm{I}_0 \mathrm{I}_1}{\gamma\alpha^2 \mathrm{I}_0^2 - \mathrm{I}_1^2 (1+\gamma\alpha^2)} \quad . \quad . \quad . \quad . \quad (94),$$

so that ζ is a known constant.

We then find, putting in for $\sum_1^\infty \frac{1}{n^4}$ its value $\pi^4/90$,

$$u_0 = -\mathrm{E} \left[\left(\frac{1-\gamma-\zeta}{12} \right) a^4 - \tfrac{1}{6} a^2c^2 + \tfrac{1}{3} (1 + \gamma - \tfrac{4}{15} \{2\gamma+1\}) c^4 \right] \quad . \quad (95).$$

If now we write

$$\left.\begin{aligned} f &= (2\gamma-1) \frac{a^4}{6} + \tfrac{1}{3} c^2a^2 - \tfrac{14}{45} (2\gamma+1) c^4 \\ g &= \tfrac{1}{12} (1-\gamma-\zeta) a^4 - \tfrac{1}{6} a^2c^2 + \tfrac{1}{3} (1 + \gamma - \tfrac{4}{15} \{2\gamma+1\}) c^4 \\ h &= -\tfrac{14}{15} c^4\gamma - \frac{a^4}{12} (2\gamma+1) + \frac{2a^2c^2\gamma}{3} \end{aligned}\right\} \quad . \quad . \quad (96),$$

so that f, g, h are known constants, then equations (90), (93), (95) may be re-written as follows:—

$$\left.\begin{aligned} 2\lambda u_0 + (\lambda + 2\mu) w_0 + \mu f \mathrm{E} &= -\mathrm{Q} \\ u_0 \qquad\qquad + \quad g\mathrm{E} &= 0 \\ 2(\lambda+\mu) u_0 + \lambda w_0 + \mu h \mathrm{E} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (97).$$

Solving, we have:

$$\mu \mathrm{E} = \frac{-\mathrm{Q}(2\gamma-1)}{(h + (2\gamma-1)f + 2g(4\gamma-1))} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98),$$

$$u_0 = \frac{\mathrm{Q}g(2\gamma-1)}{\mu(h + (2\gamma-1)f + 2g(4\gamma-1))} \quad . \quad . \quad . \quad . \quad . \quad . \quad (99),$$

$$w_0 = -\frac{Q}{\mu}\frac{h(1-\gamma)+2g\gamma}{h+(2\gamma-1)f+2g(4\gamma-1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (100).$$

All the constants are therefore absolutely determinate and the solution is complete.

§ 16. *Expressions for the Stresses.*

The reduced expressions for the four stresses are given below :

$$\widehat{rr} = \mu E \left\{ \begin{aligned} & -\left\{\frac{2\gamma+1}{12}a^4 - \frac{2\gamma}{3}a^2c^2 + \tfrac{14}{15}\gamma c^4\right\} + r^2\left\{(1+\gamma)\frac{a^2}{4} - (4\gamma+1)\frac{c^2}{3}\right\} \\ & + z^2 \times 2\gamma[2c^2 - a^2] - r^4\left(\frac{\gamma+2}{12}\right) + r^2z^2(4\gamma+1) - 2\gamma z^4 \\ & + \frac{16ac^3}{\pi^3}(2\gamma+1)\sum_1^\infty \frac{(-1)^n}{n^3 I_1(\alpha)}\left(\frac{I_1(\rho)}{\rho} - I_0(\rho)\right)\cos\frac{n\pi z}{c} \\ & - \frac{4c^2a^2\gamma}{\pi^2}\sum_1^\infty \frac{(-1)^n}{n^2}\frac{\left[\left\{(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right\}I_1 - (8\gamma+4)I_0\right]}{\gamma\alpha^2(I_0^2 - I_1^2) - I_1^2} \\ & \qquad \left[\left(\frac{\alpha I_0}{I_1}+1\right)I_0(\rho) - \left(\frac{\alpha I_0}{I_1} + \frac{1}{\gamma}\right)\frac{I_1(\rho)}{\rho} - \rho I_1(\rho)\right]\cos\frac{n\pi z}{c} \end{aligned} \right\} \quad (101).$$

$$\widehat{zz} = -Q$$

$$+\mu E \left\{ \begin{aligned} & -\left\{\frac{2\gamma-1}{6}a^4 + \tfrac{1}{3}a^2c^2 - \tfrac{14}{45}(2\gamma+1)c^4\right\} + r^2\left\{\frac{(2\gamma-1)}{2}a^2 + \frac{2c^2}{3}\right\} \\ & - z^2\left(\tfrac{4}{3}(2\gamma+1)c^2 - a^2\right) - r^4\frac{2\gamma-1}{4} - 2r^2z^2 + \tfrac{2}{3}(2\gamma+1)z^4 \\ & + \frac{16ac^3}{\pi^3}(2\gamma+1)\sum_1^\infty \frac{(-1)^n}{n^3}\frac{I_0(\rho)}{I_1(\alpha)}\cos\frac{n\pi z}{c} \\ & - \frac{4c^2a^2\gamma}{\pi^2}\sum_1^\infty \frac{(-1)^n}{n^2}\frac{\left[\left\{(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right\}I_1 - (8\gamma+4)I_0\right]}{\gamma\alpha^2(I_0^2 - I_1^2) - I_1^2} \\ & \qquad \left[\rho I_1(\rho) - I_0(\rho)\left\{\frac{\alpha I_0}{I_1} - 2\right\}\right]\cos\frac{n\pi z}{c} \end{aligned} \right\} \quad . \quad (102).$$

$$\widehat{\phi\phi} = \mu E \left\{ \begin{aligned} & -\left\{\frac{2\gamma+1}{12}a^4 - \frac{2\gamma}{3}a^2c^2 + \tfrac{14}{15}\gamma c^4\right\} + r^2\left\{(3\gamma-1)\frac{a^2}{4}\right. \\ & \qquad\qquad \left. - (4\gamma-1)\frac{c^2}{3}\right\} + 2\gamma z^2\{2c^2 - a^2\} \\ & - r^4\left(\frac{5\gamma-2}{12}\right) + r^2z^2(4\gamma-1) - 2\gamma z^4 \\ & \qquad - \frac{16ac^3}{\pi^3}(2\gamma+1)\sum_1^\infty \frac{(-1)^n}{n^3}\frac{I_1(\rho)}{\rho I_1(\alpha)}\cos\frac{n\pi z}{c} \\ & - \frac{4c^2a^2\gamma}{\pi^2}\sum_1^\infty \frac{(-1)^n}{n^2}\frac{\left[\left\{(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right\}I_1 - (8\gamma+4)I_0\right]}{\gamma\alpha^2(I_0^2 - I_1^2) - I_1^2} \\ & \qquad \left[\frac{I_1(\rho)}{\rho}\left(\frac{\alpha I_0}{I_1} + \frac{1}{\gamma}\right) - I_0(\rho)\left(\frac{1}{\gamma} - 1\right)\right]\cos\frac{n\pi z}{c} \end{aligned} \right\} \quad . \quad (103).$$

$$\widehat{rz} = \mu E \left\{ \begin{array}{l} \{\frac{4}{3}(2\gamma+1)c^2 - a^2\}rz + r^3z - \frac{4}{3}(2\gamma+1)rz^3 \\ + (2\gamma+1)\frac{16ac^3}{\pi^3}\sum_1^\infty \frac{(-1)^n}{n^3}\frac{I_1(\rho)}{I_1(\alpha)}\sin\frac{n\pi z}{c} \\ - \frac{4c^2a^2\gamma}{\pi^2}\sum_1^\infty \frac{(-1)^n}{n^2} \frac{\left[\left\{(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right\}I_1 - (8\gamma+4)I_0\right]}{\gamma\alpha^2(I_0^2 - I_1^2) - I_1^2} \\ \qquad \left[\rho I_0(\rho) - \alpha\frac{I_0}{I_1}I_1(\rho)\right]\sin\frac{n\pi z}{c} \end{array} \right\} \quad . \quad (104),$$

where, putting in for f, g, h in (98) their values,

$$E = -\frac{Q}{\mu}\frac{2\gamma - 1}{(4\gamma-1)\left(\frac{8}{45}c^4 - \frac{a^4\zeta}{6}\right) - \frac{a^4}{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105).$$

§ 17. *Numerical Example.*

Let us now consider a concrete example. Take a cylinder whose diameter is nearly equal to its length. This corresponds about to the dimensions used in practice for test pieces under pressure. We will take $\pi a/c = 3$ in order to simplify the calculation of the I-functions.

Further, we shall assume uniconstant isotropy, so that $\gamma = 2/3$.

We then find :

$$\left. \begin{array}{lll} u_0 = -\frac{Q}{\mu}(\cdot 10695), & u_1 = -\frac{Q}{\mu a^2}(\cdot 481906), & u_2 = -\frac{Q}{\mu a^4}(\cdot 079057), \\ w_0 = -\frac{Q}{\mu}(\cdot 090552), & w_1 = -\frac{Q}{\mu a^2}(1\cdot 46046), & w_2 = \frac{Q}{\mu a^4}(1\cdot 01193), \\ E = \frac{Q}{\mu a^4}(1\cdot 13842), & E' = -\frac{Q}{\mu a^4}(1\cdot 01193), & F = -\frac{Q}{\mu a^4}(2\cdot 52982), \\ & D = \frac{Q}{\mu a^2}(1\cdot 63584), & D' = \frac{Q}{\mu a^2}(1\cdot 10970), \\ & \zeta = 2\cdot 036847, & \\ f = -a^4(\cdot 451889), & g = a^4(\cdot 093947), & h = -a^4(\cdot 455329) \end{array} \right\} \quad . \quad (106),$$

and the stresses become :

$$\left.\begin{aligned}
\frac{\widehat{rr}}{\mu \mathrm{E}} &= -\,{\cdot}455329a^4 - {\cdot}923650r^2a^2 + 1{\cdot}590994z^2a^2 \\
&\quad - \tfrac{2}{9}r^4 + \tfrac{11}{3}r^2z^2 - \tfrac{4}{3}z^4 + \tfrac{112}{81}a^4\sum_1^\infty \frac{(-1)^n}{n^3}c_n \cos\frac{n\pi z}{c} \\
&\quad - \frac{8a^4}{27}\sum_1^\infty \frac{(-1)^n}{n^2}e_n \cos\frac{n\pi z}{c} \\
\frac{\widehat{zz}+\mathrm{Q}}{\mu \mathrm{E}} &= {\cdot}451889a^4 + {\cdot}897748r^2a^2 - 2{\cdot}411716z^2a^2 \\
&\quad - \frac{r^4}{12} - 2r^2z^2 + \tfrac{14}{9}z^4 + \tfrac{112}{81}a^4\sum_1^\infty \frac{(-1)^n}{n^3}f_n \cos\frac{n\pi z}{c} \\
&\quad - \frac{8a^4}{27}\sum_1^\infty \frac{(-1)^n}{n^2}g_n \cos\frac{n\pi z}{c} \\
\frac{\widehat{\phi\phi}}{\mu \mathrm{E}} &= -\,{\cdot}455329a^4 - {\cdot}359235r^2a^2 + 1{\cdot}590994z^2a^2 - \frac{r^4}{9} \\
&\quad + \frac{5r^2z^2}{3} - \tfrac{4}{3}z^4 - \tfrac{112}{81}a^4\sum_1^\infty \frac{(-1)^n}{n^3}h_n \cos\frac{n\pi z}{c} - \frac{8a^4}{27}\sum_1^\infty \frac{(-1)^n}{n^2}l_n \cos\frac{n\pi z}{c} \\
\frac{\widehat{rz}}{\mu \mathrm{E}} &= 2{\cdot}411716a^2rz + r^3z - \tfrac{28}{9}rz^3 + \tfrac{112}{81}a^4\sum_1^\infty \frac{(-1)^n}{n^3}p_n \sin\frac{n\pi z}{c} \\
&\quad - \tfrac{8}{27}a^4\sum_1^\infty \frac{(-1)^n}{n^2}q_n \sin\frac{n\pi z}{c}
\end{aligned}\right\} \quad . \quad . \quad (106a),$$

where

$$\left.\begin{aligned}
c_n &= \left\{\frac{\mathrm{I}_1(\rho)}{\rho} - \mathrm{I}_0(\rho)\right\} \Big/ \mathrm{I}_1(\alpha) \\
e_n &= \frac{\left[\left(7n + \dfrac{76}{9n}\right)\mathrm{I}_1(\alpha) - \tfrac{28}{3}\mathrm{I}_0(\alpha)\right]\left[\left(\dfrac{\alpha \mathrm{I}_0}{\mathrm{I}_1} + 1\right)\mathrm{I}_0(\rho) - \left(\dfrac{\alpha\mathrm{I}_0}{\mathrm{I}_1} + \dfrac{1}{\gamma}\right)\dfrac{\mathrm{I}_1(\rho)}{\rho} - \mathrm{I}_1(\rho)\right]}{6n^2(\mathrm{I}_0^2 - \mathrm{I}_1^2) - \mathrm{I}_1^2} \\
f_n &= \mathrm{I}_0(\rho) / \mathrm{I}_1(\alpha) \\
g_n &= \frac{\left[\left(7n + \dfrac{76}{9n}\right)\mathrm{I}_1 - \tfrac{28}{3}\mathrm{I}_0\right]\left[\rho\mathrm{I}_1(\rho) - \mathrm{I}_0(\rho)\left\{\dfrac{\alpha\mathrm{I}_0}{\mathrm{I}_1} - 2\right\}\right]}{6n^2(\mathrm{I}_0^2 - \mathrm{I}_1^2) - \mathrm{I}_1^2} \\
h_n &= \mathrm{I}_1(\rho) / \rho\mathrm{I}_1(\alpha) \\
l_n &= \frac{\left[\left(7n + \dfrac{76}{9n}\right)\mathrm{I}_1 - \tfrac{28}{3}\mathrm{I}_0\right]\left[\dfrac{\mathrm{I}_1(\rho)}{\rho}\left(\dfrac{\alpha\mathrm{I}_0}{\mathrm{I}_1} + \dfrac{1}{\gamma}\right) - \left(\dfrac{1}{\gamma} - 1\right)\mathrm{I}_0(\rho)\right]}{6n^2(\mathrm{I}_0^2 - \mathrm{I}_1^2) - \mathrm{I}_1^2} \\
p_n &= \mathrm{I}_1(\rho)/\mathrm{I}_1(\alpha) \\
q_n &= \frac{\left[\left(7n + \dfrac{76}{9n}\right)\mathrm{I}_1 - \tfrac{28}{3}\mathrm{I}_0\right]\left[\rho\mathrm{I}_0(\rho) - \dfrac{\alpha\mathrm{I}_0}{\mathrm{I}_1}\mathrm{I}_1(\rho)\right]}{6n^2(\mathrm{I}_0^2 - \mathrm{I}_1^2) - \mathrm{I}_1^2}
\end{aligned}\right\} \quad . \quad (106b)$$

§ 18. *Tables of the Constants for the special case taken.*

The values of these constants I have calculated for the values 1 to 6 of n and the values $\rho = 0$, $\alpha/3$, $2\alpha/3$, α, *i.e.*, remembering that in our case $\alpha = 3n$, for $\rho = 0$, n, $2n$, $3n$. These values are given in the following tables:—

Table of Constants.

c_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	$-\cdot 126474$	$-\cdot 177294$	$-\cdot 375444$	$-\cdot 901257$
2	$-\cdot 815103 \times 10^{-2}$	$-\cdot 241966 \times 10^{-1}$	$-\cdot 144470$	$-\cdot 929393$
3	$-\cdot 485003 \times 10^{-3}$	$-\cdot 345613 \times 10^{-2}$	$-\cdot 553007 \times 10^{-1}$	$-\cdot 949670$
4	$-\cdot 275613 \times 10^{-4}$	$-\cdot 488500 \times 10^{-3}$	$-\cdot 208129 \times 10^{-1}$	$-\cdot 961180$
5	$-\cdot 152381 \times 10^{-5}$	$-\cdot 681835 \times 10^{-4}$	$-\cdot 776521 \times 10^{-2}$	$-\cdot 968455$
6	$-\cdot 825385 \times 10^{-7}$	$-\cdot 943397 \times 10^{-5}$	$-\cdot 288544 \times 10^{-2}$	$-\cdot 973449$

e_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	$\cdot 971897$	$1\cdot 132478$	$1\cdot 573394$	$1\cdot 960798$
2	$\cdot 120292$	$\cdot 260646$	$\cdot 914018$	$1\cdot 998081$
3	$\cdot 117631 \times 10^{-1}$	$\cdot 586808 \times 10^{-1}$	$\cdot 534190$	$2\cdot 319025$
4	$\cdot 945472 \times 10^{-3}$	$\cdot 115601 \times 10^{-1}$	$\cdot 273635$	$2\cdot 545291$
5	$\cdot 678980 \times 10^{-4}$	$\cdot 208131 \times 10^{-2}$	$\cdot 129351$	$2\cdot 702777$
6	$\cdot 454266 \times 10^{-5}$	$\cdot 353345 \times 10^{-3}$	$\cdot 582489 \times 10^{-1}$	$2\cdot 816948$

f_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	$\cdot 252949$	$\cdot 320250$	$\cdot 576618$	$1\cdot 234590$
2	$\cdot 163021 \times 10^{-1}$	$\cdot 371619 \times 10^{-1}$	$\cdot 184245$	$1\cdot 096059$
3	$\cdot 970005 \times 10^{-3}$	$\cdot 473439 \times 10^{-2}$	$\cdot 652177 \times 10^{-1}$	$1\cdot 060779$
4	$\cdot 551226 \times 10^{-4}$	$\cdot 622991 \times 10^{-3}$	$\cdot 235682 \times 10^{-1}$	$1\cdot 044513$
5	$\cdot 304762 \times 10^{-5}$	$\cdot 830167 \times 10^{-4}$	$\cdot 857923 \times 10^{-2}$	$1\cdot 035120$
6	$\cdot 165477 \times 10^{-6}$	$\cdot 111258 \times 10^{-4}$	$\cdot 313561 \times 10^{-2}$	$1\cdot 029005$

g_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	$-\cdot 787811$	$-\cdot 736099$	$-\cdot 324884$	$1\cdot 638888$
2	$-\cdot 155588$	$-\cdot 246518$	$-\cdot 431227$	$2\cdot 052233$
3	$-\cdot 176721 \times 10^{-1}$	$-\cdot 584824 \times 10^{-1}$	$-\cdot 326343$	$2\cdot 400197$
4	$-\cdot 152827 \times 10^{-2}$	$-\cdot 116088 \times 10^{-1}$	$-\cdot 189318$	$2\cdot 623837$
5	$-\cdot 114613 \times 10^{-3}$	$-\cdot 209107 \times 10^{-2}$	$-\cdot 963285 \times 10^{-1}$	$2\cdot 774995$
6	$-\cdot 789128 \times 10^{-5}$	$-\cdot 354776 \times 10^{-3}$	$-\cdot 455547 \times 10^{-1}$	$2\cdot 882627$

h_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	$\cdot 126474$	$\cdot 142956$	$\cdot 201175$	$\cdot 333333$
2	$\cdot 815103 \times 10^{-2}$	$\cdot 129653 \times 10^{-1}$	$\cdot 397748 \times 10^{-1}$	$\cdot 166667$
3	$\cdot 485003 \times 10^{-3}$	$\cdot 127826 \times 10^{-2}$	$\cdot 991703 \times 10^{-2}$	$\cdot 111111$
4	$\cdot 275613 \times 10^{-4}$	$\cdot 134492 \times 10^{-3}$	$\cdot 275530 \times 10^{-2}$	$\cdot 853333 \times 10^{-1}$
5	$\cdot 152381 \times 10^{-5}$	$\cdot 148331 \times 10^{-4}$	$\cdot 814018 \times 10^{-3}$	$\cdot 666667 \times 10^{-1}$
6	$\cdot 827385 \times 10^{-7}$	$\cdot 169178 \times 10^{-5}$	$\cdot 250165 \times 10^{-3}$	$\cdot 555556 \times 10^{-1}$

l_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	$\cdot 971897$	$1\cdot 067168$	$1\cdot 386651$	$2\cdot 042427$
2	$\cdot 120292$	$\cdot 179629$	$\cdot 477822$	$1\cdot 664308$
3	$\cdot 117631 \times 10^{-1}$	$\cdot 283739 \times 10^{-1}$	$\cdot 185745$	$1\cdot 682707$
4	$\cdot 945472 \times 10^{-3}$	$\cdot 414783 \times 10^{-2}$	$\cdot 707563 \times 10^{-1}$	$1\cdot 703514$
5	$\cdot 678980 \times 10^{-4}$	$\cdot 586774 \times 10^{-3}$	$\cdot 266082 \times 10^{-1}$	$1\cdot 716955$
6	$\cdot 454266 \times 10^{-5}$	$\cdot 817120 \times 10^{-4}$	$\cdot 993188 \times 10^{-2}$	$1\cdot 725546$

p_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	$\cdot 000000$	$\cdot 142956$	$\cdot 402350$	$1\cdot 000000$
2	$\cdot 000000$	$\cdot 259307 \times 10^{-1}$	$\cdot 159100$	$1\cdot 000000$
3	$\cdot 000000$	$\cdot 383479 \times 10^{-2}$	$\cdot 595020 \times 10^{-1}$	$1\cdot 000000$
4	$\cdot 000000$	$\cdot 537967 \times 10^{-3}$	$\cdot 220423 \times 10^{-1}$	$1\cdot 000000$
5	$\cdot 000000$	$\cdot 741658 \times 10^{-4}$	$\cdot 814019 \times 10^{-2}$	$1\cdot 000000$
6	$\cdot 000000$	$\cdot 101507 \times 10^{-4}$	$\cdot 300198 \times 10^{-2}$	$1\cdot 000000$

q_n.

n.	$r = 0$.	$r = a/3$.	$r = 2a/3$.	$r = a$.
1	·000000	− ·382470	− ·615993	·000000
2	·000000	− ·200638	− ·645084	·000000
3	·000000	− ·540924 × 10^{-1}	− ·426702	·000000
4	·000000	− ·111882 × 10^{-1}	− ·230912	·000000
5	·000000	− ·204756 × 10^{-2}	− ·112874	·000000
6	·000000	− ·349986 × 10^{-3}	− ·518839 × 10^{-1}	·000000

The above, when substituted in the formulæ, give quite fairly rapid convergence when $r < a$, the convergency ratio being in this case less than unity by a finite amount. But when $r = a$, the series becomes comparable with the series $\Sigma \frac{(-1)^n}{n^i} \cos \frac{n\pi z}{c}$, where i is a positive integer, and, as in the first problem, a special procedure has to be adopted.

§ 19. *Methods of Evaluation at the Curved Boundary.*

When $r = a$, $\widehat{rr}$ and $\widehat{rz}$ are of course zero; but the stresses $\widehat{zz}$ and $\widehat{\phi\phi}$ require separate evalution.

Now, if we use the series for I_0 and I_1 in descending powers of the argument, the first few terms of these give a very good representation of the function when α is at all large, and will be quite sufficient for $\alpha > 18$, at which point the tables of the last paragraph stop.

Replacing I_0, I_1 by these series, we find

$$\left.\begin{aligned} f_n(\alpha) &= 1 + \frac{1}{2\alpha} + \frac{3}{8\alpha^2} + \frac{3}{8\alpha^3} + \dots \\ c_n(\alpha) &= -\left(1 - \frac{1}{2\alpha} + \frac{3}{8\alpha^2} + \frac{3}{8\alpha^3} + \dots\right) \\ l_n(\alpha) &= \tfrac{7}{4} - \frac{165}{16\alpha^2} + \frac{335}{8\alpha^3} + \dots \\ e_n(\alpha) &= \tfrac{7}{2} - \frac{14}{\alpha} + \frac{31}{\alpha^2} - \frac{21}{4\alpha^3} - \frac{21}{4\alpha^4} + \dots \\ g_n(\alpha) &= \tfrac{7}{2} - \frac{49}{4\alpha} + \frac{157}{8\alpha^2} + \frac{503}{32\alpha^3} + \dots \end{aligned}\right\} \quad \dots \dots (107),$$

and

$$h\ (\alpha) = \frac{1}{\alpha}, \quad p_n(\alpha) = 1, \quad q_n(\alpha) = 0.$$

Now write

$$l_n'(\alpha) = -l_n^{(4)}(\alpha) + l_n(\alpha),$$
$$g_n'(\alpha) = -g_n^{(4)}(\alpha) + g_n(\alpha),$$
$$f_n'(\alpha) = -f_n^{(4)}(\alpha) + f_n(\alpha),$$

where $l_n^{(4)}(\alpha)$ denotes the first four terms of the above series for $l_n(\alpha)$, with a similar meaning for $g_n^{4}(\alpha)$ and $f_n^{(4)}(\alpha)$.

Then if we substitute in the equations for $\widehat{zz} + Q, \widehat{\phi\phi}$ we find:

$$\frac{(\widehat{zz} + Q)_{r=a}}{\mu E} = 1{\cdot}266304\, a^4 - 4{\cdot}411716\, a^2 z^2 + \tfrac{14}{9} z^4$$

$$+ a^4 \times \tfrac{8}{27}\left\{ -\tfrac{7}{2}\sum_1^\infty \frac{(-1)^n}{n^2}\cos\frac{n\pi z}{c} - \tfrac{101}{72}\sum_1^\infty \frac{(-1)^n}{n^4}\cos\frac{n\pi z}{c} + \tfrac{7}{108}\sum_1^\infty \frac{(-1)^n}{n^6}\cos\frac{n\pi z}{c}\right\}$$

$$+ \frac{8a^4}{27}\sum_1^\infty (-1)^n \cos\frac{n\pi z}{c}\left\{\tfrac{14}{3}\frac{f_n'}{n^3} - \frac{g_n'}{n^2}\right\} + \frac{8a^4}{27}\sum_1^\infty (-1)^n \cos\frac{n\pi z}{c}\left\{\tfrac{35}{4}\frac{1}{n^3} - \tfrac{335}{864}\frac{1}{n^5}\right\}.$$

Now remembering that

$$\sum_1^\infty \frac{(-1)^n}{n^2}\cos\frac{n\pi z}{c} = \frac{\pi^2}{4}\frac{z^2}{c^2} - \frac{\pi^2}{12}$$

$$\sum_1^\infty \frac{(-1)^n}{n^4}\cos\frac{n\pi z}{c} = -\frac{z^4}{c^4}\frac{\pi^4}{48} + \frac{z^2}{c^2}\frac{\pi^4}{24} - \frac{7\pi^4}{720}$$

$$\sum_1^\infty \frac{(-1)^n}{n^6}\cos\frac{n\pi z}{c} = \frac{z^6}{c^6}\frac{\pi^6}{1440} - \frac{z^4}{c^4}\frac{\pi^6}{288} + \frac{z^2}{c^2}\frac{7\pi^6}{1440} - \frac{31\pi^6}{30240},$$

we find, using $\pi a/c = 3$,

$$\frac{(\widehat{zz} + Q)_{r=a}}{\mu E a^4} = 2{\cdot}493930$$
$$- 8{\cdot}201525\frac{z^2}{a^2} + 2{\cdot}251614\frac{z^4}{a^4} + {\cdot}00972222\frac{z^6}{a^6}$$
$$+ \sum_1^\infty (-1)^n \cos\frac{n\pi z}{c}\left\{\tfrac{112}{81}\frac{f_n'}{n^3} - \tfrac{8}{27}\frac{g_n'}{n^2} + \tfrac{70}{27}\frac{1}{n^3} - \tfrac{335}{2916}\frac{1}{n^5}\right\} \quad . \quad . \quad . \quad (108).$$

And, in a similar manner,

$$\frac{(\widehat{\phi\phi})_{r=a}}{\mu E a^4} = -{\cdot}3842415 + 1{\cdot}6416832\frac{z^2}{a^2} - 1{\cdot}1284672\frac{z^4}{a^4}$$
$$- \sum_1^\infty (-1)^n \cos\frac{n\pi z}{c}\left\{\tfrac{8}{27}\frac{l_n'}{n^2} + \tfrac{335}{729}\frac{1}{n^5}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109).$$

§ 20. *Calculation of the Series in the preceding Section.*

If we work out the values of f_n', g_n', l_n' we find they are as tabulated below :

n.	f_n'.	g_n'.	l_n'.
1	+ ·012367	− ·540511	− ·112667
2	+ ·000573	− ·024011	+ ·006901
3	+ ·000078	− ·002538	+ ·002580
4	+ ·000025	− ·001259	+ ·000896
5	+ ·000009	− ·000217	+ ·000381
6	+ ·000005	− ·000083	+ ·000195

Hence the parts of the Σ which depend upon l_n'/n^2, f_n'/n^3, g_n'/n^2 converge quite rapidly enough to allow us to stop after the sixth term. It therefore merely remains to evaluate the series

$$\sum_1^\infty \frac{(-1)^n}{n^3} \cos \frac{n\pi z}{c} \quad \text{and} \quad \sum_1^\infty \frac{(-1)^n}{n^5} \cos \frac{n\pi z}{c}.$$

These cannot be expressed in finite terms, and although we may apply the EULER-MACLAURIN sum-formula to these series directly, though in a slightly modified form, this sum-formula is not really of very great advantage, as its rapidity of convergence depends on z, and is such, for certain values of this variable, as to render the formula useless as an approximation to the remainder. As a matter of fact, however, the series were to be calculated only for values of $z = ic/6$, i being any integer from 0 to 6. But for such values of z, the cosine terms repeat themselves after $n = 6$.

Thus,

$$\sum_1^\infty \frac{(-1)^{n-1}}{n^3} \cos\left(\frac{n\pi i}{6}\right)$$

$$= \cos i\pi/6 \left(\sum_{m=0}^{m=\infty} \frac{1}{(1+12m)^3} + (-1)^i \sum_{m=0}^{m=\infty} \frac{1}{(7+12m)^3}\right)$$

$$- \cos 2i\pi/6 \left(\sum_{m=0}^{m=\infty} \frac{1}{(2+12m)^3} + (-1)^i \sum_{m=0}^{m=\infty} \frac{1}{(8+12m)^3}\right)$$

$$+ \ldots\ldots$$

$$- \cos i\pi \left(\sum_{m=0}^{m=\infty} \frac{1}{(6+12m)^3} + (-1)^i \sum_{m=0}^{m=\infty} \frac{1}{(12+12m)^3}\right).$$

A precisely similar formula holds for

$$\sum_1^\infty \frac{(-1)^{n-1}}{n^5} \cos\left(\frac{n\pi i}{6}\right).$$

Thus we see we need only work out the series

$$\sum_{m=0}^{m=\infty} \frac{1}{(s+12m)^3} \quad \text{and} \quad \sum_{m=0}^{m=\infty} \frac{1}{(s+12m)^5},$$

where s has integral values ranging from 1 to 12.

These series are easily calculated, and, to them, the sum-formula is quite applicable.

By this means it was found that

$$\sum_1^\infty \frac{(-1)^{n-1}}{n^5} \cos\left(\frac{n\pi i}{6}\right)$$

$$\begin{aligned}
&= \cos\frac{i\pi}{6} \quad (1\cdot000{,}0027 + (-1)^i \cdot000{,}0598) \\
&- \cos 2i\pi/6 \; (\cdot031{,}2519 + (-1)^i \cdot000{,}0308) \\
&+ \cos 3i\pi/6 \; (\cdot004{,}1165 + (-1)^i \cdot000{,}0171) \\
&- \cos 4i\pi/6 \; (\cdot000{,}9776 + (-1)^i \cdot000{,}0102) \\
&+ \cos 5i\pi/6 \; (\cdot000{,}3207 + (-1)^i \cdot000{,}0064) \\
&- \cos i\pi \quad (\cdot000{,}1291 + (-1)^i \cdot000{,}0041)
\end{aligned}$$

and $$\sum_1^\infty \frac{(-1)^{n-1}}{n^3} \cos\left(\frac{n\pi i}{6}\right)$$

$$\begin{aligned}
&= \cos i\pi/6 \; (1\cdot000{,}5611 + (-1)^i \cdot003{,}1246) \\
&- \cos 2i\pi/6 \; (\cdot125{,}4607 + (-1)^i \cdot002{,}1368) \\
&+ \cos 3i\pi/6 \; (\cdot037{,}4212 + (-1)^i \cdot001{,}5343) \\
&- \cos 4i\pi/6 \; (\cdot015{,}9496 + (-1)^i \cdot001{,}1448) \\
&+ \cos 5i\pi/6 \; (\cdot008{,}2777 + (-1)^i \cdot000{,}8811) \\
&- \cos i\pi \quad (\cdot004{,}8694 + (-1)^i \cdot000{,}6956),
\end{aligned}$$

and the calculation of the stresses on the boundary then became a simple matter.

§ 21. *Numerical Values of the Stresses.*

The values of the stresses, referred to the mean pressure as unit, are tabulated below :

TABLE of Stresses.

$\widehat{zz}$/Q.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	1·13382	−1·13436	−1·13322	−1·12145	−1·08000	−1·03372	− ·68576
$a/3$	−1·09971	−1·10053	−1·10017	−1·08998	−1·05441	− ·96416	− ·74952
$2a/3$	−1·00724	−1·00896	−1·01314	−1·01553	−1·00557	− ·97136	− ·92845
$5a/6$	− ·94841	− ·94843	− ·95056	− ·96037	− ·98465	−1·01726	−1·08211
a	− ·89430	− ·88809	− ·87216	− ·85845	− ·88177	−1·04077	−1·68635

$\widehat{rr}$/Q.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	− ·00274	− ·01414	− ·05134	− ·12416	− ·25325	− ·48151	− ·89668
$a/3$	− ·00181	− ·01118	− ·04201	− ·10220	− ·20471	− ·37182	− ·65935
$2a/3$	− ·00110	− ·00459	− ·01759	− ·04481	− ·09244	− ·14271	− ·09977
a	·00000	·00000	·00000	·00000	·00000	·00000	·00000

$\widehat{\phi\phi}$/Q.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	− ·00274	− ·01414	− ·05134	− ·12416	− ·25325	− ·48151	− ·89668
$a/3$	·00299	− ·00737	− ·04126	− ·10748	− ·22288	− ·42120	− ·77728
$2a/3$	·01744	·01017	− ·01461	− ·06480	− ·15199	− ·28607	− ·48160
a	·03265	·03013	·02030	− ·00516	− ·06344	− ·18728	− ·43801

$\widehat{rz}$/Q.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	·00000	·00000	·00000	·00000	·00000	·00000	·00000
$a/3$	·00000	·00705	− ·00388	− ·02020	− ·05902	− ·14798	− ·35357
$2a/3$	·00000	·00425	− ·00171	− ·02712	− ·08645	− ·19750	− ·44154
a	·00000	·00000	·00000	·00000	·00000	·00000	·00000

In the above the values of $\widehat{zz}$ for an additional value of r (viz., $r = 5a/6$) have been computed in order to exhibit more clearly the variation in the pressure along the radius.

The numerical results here given are shown graphically in Diagrams 7–10. We see at once that, save near the ends, the stresses $\widehat{rr}$, $\widehat{\phi\phi}$, and $\widehat{rz}$ do not differ very much from zero, which is the value they should have on the uniform pressure hypothesis. On the other hand, the axial pressure deviates throughout from uniformity over the cross-section, the total variation in any section remaining tolerably constant over nearly two-thirds of the length of the cylinder, and equal to about 25 per cent. of the mean pressure.

Diagram 7.—Showing Stress $\widehat{zz}$ for Cylinder compressed between Rough Planes (second example).

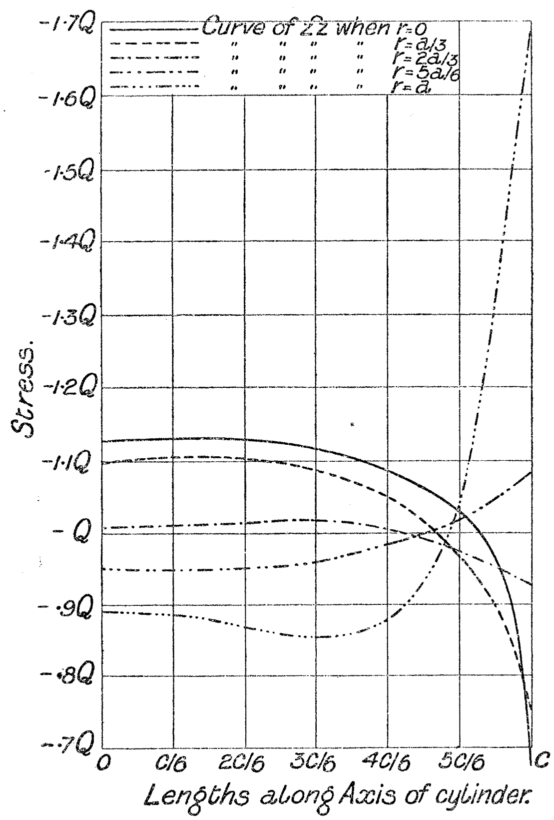
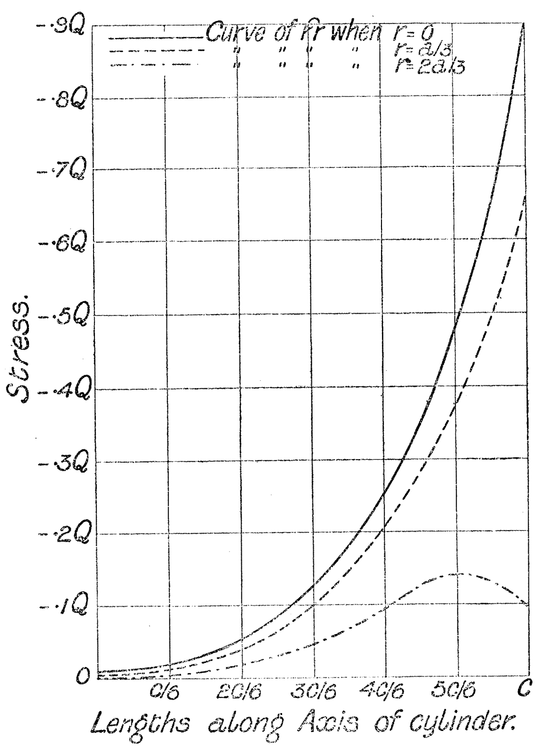


Diagram 8.—Showing Stress $\widehat{rr}$ for Cylinder compressed between Rough Planes (second example).



We notice also, that, near the centre of the cylinder, the greatest pressures occur at the centre of the cross-section; whereas the reverse takes place at the ends, the pressure at the perimeter of the ends amounting to about $1\frac{2}{3}$ times the mean pressure.

If we bear in mind the suggestion of the first problem of the present paper, that surface shear depresses those parts of the material towards which it acts, it is easy to see, physically, why such a distribution of pressure should be expected in practice.

The system of frictional shears required to prevent the ends expanding will be towards the centre: the parts of the material round the centre will therefore be depressed, and the compressing planes (supposed rigid) will have to compress the outer portions more than the inner, if the cross-section is to retain its original plane form, *i.e.*, remain in contact with the compressing planes throughout. It is thus not surprising that the greatest pressure should be at the perimeter, being in fact nearly $2\frac{1}{2}$ times the pressure at the centre.

Diagram 9.—Showing Stress $\widehat{\phi\phi}$ for Cylinder compressed between Rough Planes (second example).

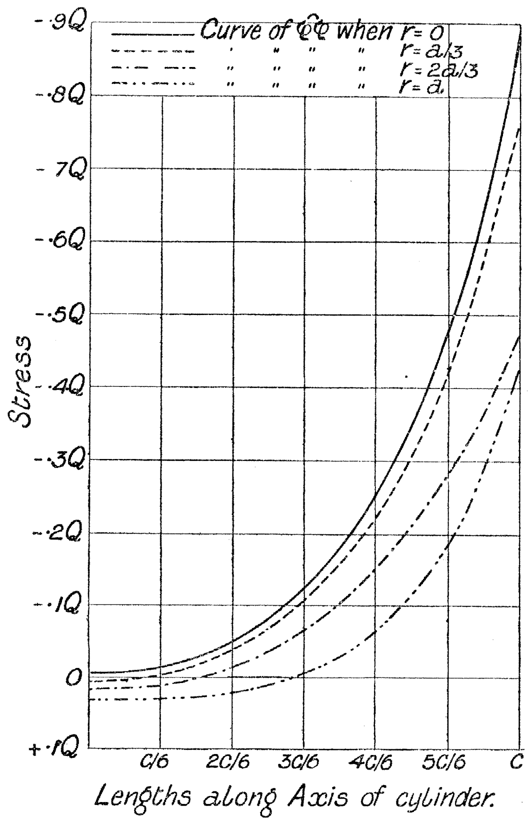
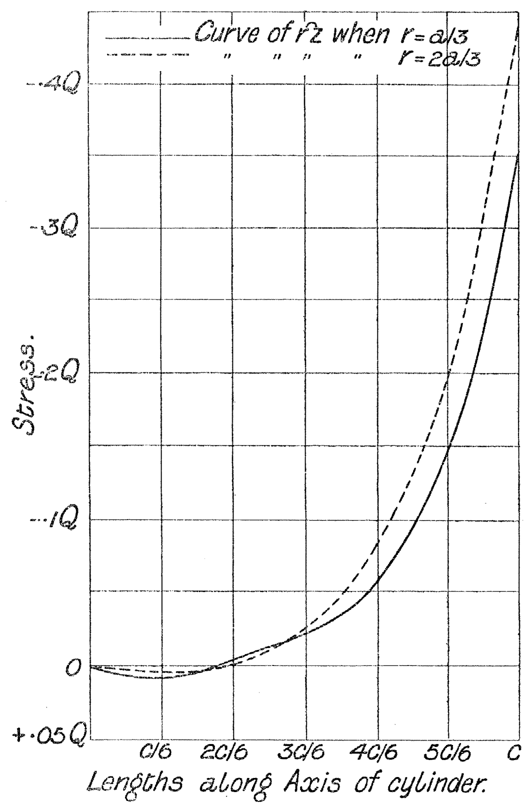


Diagram 10.—Showing Stress $\widehat{rz}$ for Cylinder compressed between Rough Planes (second example).



We see also that, near the mid-sections, the cross-radial traction changes from a pressure to a tension as we go towards the circumference, so that the outer parts of the material are subject to two pressures, parallel to the axis and the radius respectively and to a tension, at right angles to these. It should be noticed here that, in the diagrams, the ordinates representing the stresses increase negatively upwards. This has been found convenient in this case, where, owing to the general predominance of pressures, the greater part of the stresses have the negative sign.

§ 22. *Principal Stresses at each Point. Lines of Principal Stress.*

Now, when we have a distribution of stress

$$\widehat{rr},\ \widehat{\phi\phi},\ \widehat{zz},\ \widehat{rz},$$

which is symmetrical about an axis, then the principal stresses are

$$\widehat{\mathrm{RR}},\ \widehat{\phi\phi},\ \widehat{\mathrm{ZZ}};$$

where $\widehat{\phi\phi}$ is the same as before, and $\widehat{\mathrm{RR}}$, $\widehat{\mathrm{ZZ}}$ are two tractions in the meridian plane, $\widehat{\mathrm{RR}}$ making an angle θ with $\widehat{rr}$, where

$$\tan 2\theta = \frac{2\widehat{zr}}{\widehat{rr} - \widehat{zz}} \quad \text{. . . . . (110),}$$

and the values of $\widehat{\mathrm{RR}}$ and $\widehat{\mathrm{ZZ}}$ are given by

$$\left.\begin{aligned} \widehat{\mathrm{RR}} &= \frac{\widehat{rr} + \widehat{zz}}{2} + \tfrac{1}{2}\sqrt{(\widehat{rr} - \widehat{zz})^2 + 4(\widehat{zr})^2} \\ \widehat{\mathrm{ZZ}} &= \frac{\widehat{rr} + \widehat{zz}}{2} - \tfrac{1}{2}\sqrt{(\widehat{rr} - \widehat{zz})^2 + 4(\widehat{zr})^2} \end{aligned}\right\} \quad \text{. . . . . (111).}$$

Whence, using the tables in § 21, we find the following values for $\widehat{\mathrm{RR}}$ and $\widehat{\mathrm{ZZ}}$, compared with the mean pressure over the ends.

TABLE of Principal Stresses.

$\widehat{\mathrm{RR}}/\mathrm{Q}$.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	− ·00274	− ·01414	− ·05134	− ·12416	− ·25325	− ·48151	− ·89668
$a/3$	− ·00181	− ·01113	− ·04199	− ·10178	− ·20063	− ·33690	− ·34800
$2a/3$	− ·00110	− ·00457	− ·01758	− ·04405	− ·08433	− ·09804	·09139
a	·00000	·00000	·00000	·00000	·00000	·00000	·00000

$\widehat{\mathrm{ZZ}}/\mathrm{Q}$.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	− 1·13382	− 1·13436	− 1·13322	− 1·12146	− 1·08000	− 1·03372	− 0·68576
$a/3$	− 1·09971	− 1·10057	− 1·10018	− 1·09039	− 1·05849	− 0·99907	− 1·06087
$2a/3$	− 1·00724	− 1·00898	− 1·01314	− 1·01628	− 1·01467	− 1·01602	− 1·11961
a	− 0·89430	− 0·88809	− 0·87216	− 0·85845	− 0·88177	− 1·04077	− 1·68635

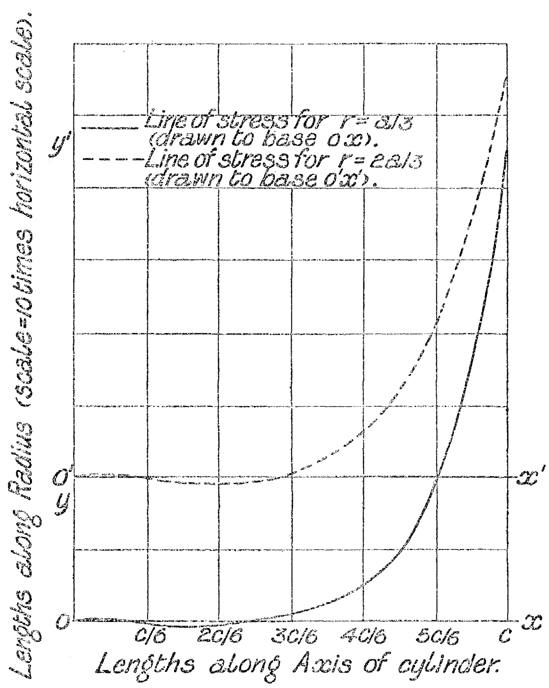
The values of tan θ for the same points are given in the following table :—

TABLE of Slope of Lines of Principal Stress (tan θ).

r.	$z = 0$.	$z = c/6$.	$z = 2z/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	0	0	0	0	0	0	0
$a/3$	0	·0064	− ·0038	− ·0204	− ·0690	− ·2358	− ·8806
$2a/3$	0	·0044	− ·0017	− ·0279	− ·0936	− ·2303	− ·4331
a	0	0	0	0	0	0	0

By the aid of the above, we may draw the lines of principal stress, which is done in Diagram 11, the slope being exaggerated in the ratio 10 : 1. In order to do so, we suppose the line of principal stress to remain always in the neighbourhood of the same generator, so that, in the above table, the values in any row apply to the same

Diagram 11.—Lines of principal Stress for Cylinder compressed between Rough Planes (second example)



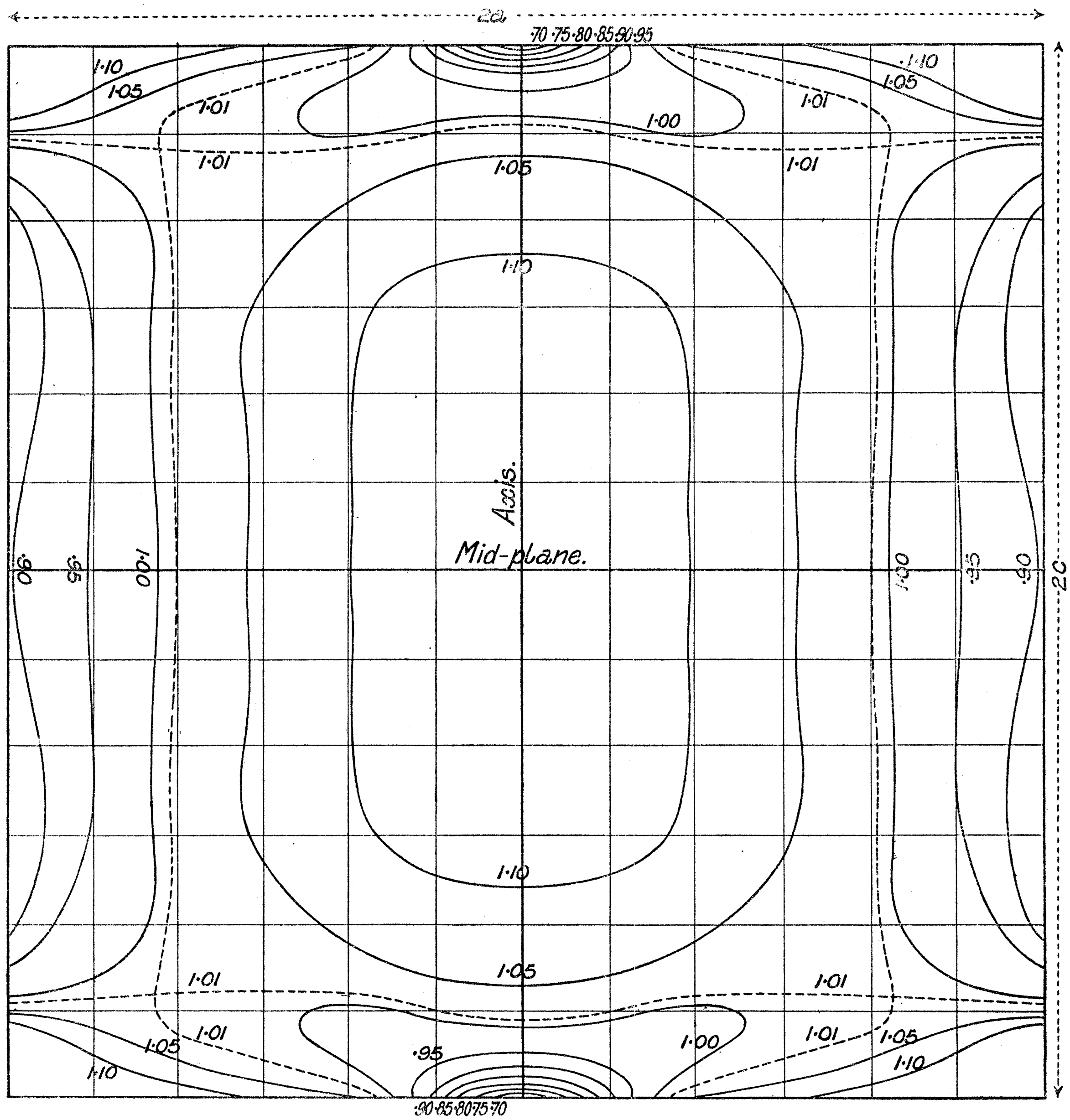
line of principal stress. This of course is not correct near the ends, but it is sufficient for our purpose, the diagram being merely intended to show the general course of the lines of stress. These are sensibly parallel to the generators throughout the middle part of the cylinder, but slope outwards near the ends.

Diagram 12* shows the distribution of the lines of equal principal stress $\widehat{ZZ}$

* In Diagram 12, as in several others, a is represented as having the same value as c, so that the horizontal and vertical scales are not strictly the same, but differ in the ratio $\pi/3$. This has been done for convenience in plotting.

inside the cylinder. I have chosen the stress $\widehat{ZZ}$ in preference to the others, as, except in the neighbourhood of the centres of the plane ends, it is everywhere the

Diagram 12.—Distribution of Principal Stress $\widehat{ZZ}$ inside the Cylinder (case of compression between Rough Rigid Planes).



The lines are drawn at intervals of ·05 Q of the stress. The critical line corresponds to $\widehat{ZZ} = 1{\cdot}01$ Q nearly

greatest of the three principal stresses, and therefore $\widehat{ZZ}$ will be the important quantity when we come to discuss the maximum stress.

The diagram has been constructed by careful interpolation from the values tabulated above, and from it we see that the surfaces of equal principal stress in the cylinder are in general made up of three sheets, and they fall into two classes: (a) those for which $\widehat{ZZ}$ has a value less than a certain critical value, which, as nearly as I can find out from graphical methods, is about 1·01 Q, and (b) those for which $\widehat{ZZ}$ has a value greater than 1·01 Q.

The surfaces (a) consist of two solid caps or buttons, round the centres of the end sections, together with a hollow cylindrical shell surrounding the middle of the cylinder. For values sensibly $<$ ·9 Q the latter sheet disappears, and only the caps remain, their volume gradually dwindling down to zero as $\widehat{ZZ}$ falls to ·686 Q.

The surfaces (b) consist of an elongated core, resembling a cylinder closed by curved ends, surrounding the centre of the compressed block, together with two annuli at the ends, as shown in the figure referred to.

The critical surface $\widehat{ZZ}$ 1·01 Q consists of two nearly plane sheets, roughly coinciding with the cross-sections $Z = \pm 5\,c/6$, and one cylindrical sheet, which bends inwards towards the end, though without completely closing in, and which roughly coincides with the cylinder $r = 2a/3$ over the greater part of its surface.

§ 23. *Application to Rupture. Distribution of Maximum Stress, Strain, and Stress Difference.*

In considering what happens when a material breaks, we have to ask, first of all, whether it be brittle or ductile. In the first case, the law of stress to strain will be approximately linear up to the point where rupture takes place; in the second case, the stress-strain relation remains approximately linear until a point is reached (called the yield-point) at which a large and sudden change occurs in the stress-strain curve, after which the material becomes sensibly plastic, so that rupture finally takes place after a large permanent deformation.

In applying an elastic theory to practice, we can, in strictness, treat of rupture only in the case of a brittle solid. Even then it has to be borne in mind that the mathematical theory of strains—upon which the equations of elasticity depend—requires the strains to be so small that their squares are negligible. It is possible that, even in the case of the most brittle solids known, this condition may cease to hold before rupture occurs, although the stress-strain relation may continue to be linear. Nevertheless, the calculated values of even the breaking strains in a material like cast iron, for instance, are so small as to render this unlikely.

For a ductile metal, such as mild steel, the elastic results only tell you where the material will begin to take permanent set.

In the case of stone or cement, however, to which the present results would be applied, there seems to be no definite yield-point or elastic limit, the material being, in fact, only imperfectly elastic throughout. Still, we may consider that the results of elastic theory give in such a case an indication of the state of stress when the specimen breaks.

There are three distinct theories, both of rupture and of failure of elasticity. According to LAMÉ and NAVIER, failure occurs when the greatest stress at any point exceeds a certain limiting value. This is also often taken as the criterion of absolute rupture. According to SAINT-VENANT, the maximum strain, and not the maximum stress, is that which determines failure. Finally, a theory has lately been put forward by various elasticians to the effect that failure occurs when the greatest shear at any point, that is, the greatest principal stress-difference, exceeds a definite amount.

Professor PERRY has proposed another criterion, suggested by the angle at which compressed cylinders shear (see 'Applied Mechanics,' pp. 344–345), namely, that rupture takes place when $s - \mu p$ exceeds a certain value, where s is the shear across any element of area at a point, p is the normal pressure on this element of area, and μ is a constant depending on the material. This theory, however, need not concern us so much, as it appears more specially applicable to the final breakdown of ductile materials long after they have become plastic. On the other hand, it has been shown by Mr. J. J. GUEST ('Phil. Mag.,' July, 1900) that the beginning of plasticity was very probably determined by the maximum stress-difference.

Let us now proceed to apply these three criteria, namely, those of the maximum stress, maximum strain, and maximum stress-difference to the cylinder in the present example, and see what results they give us, on the hypothesis that for materials like stone and cement, plastic yielding and rupture are simultaneous.

Consider first the greatest stress theory. This would make failure of elasticity first begin to occur round the perimeter of the plane ends, and that as soon as $1{\cdot}68635\,Q >$ a certain limiting value S_0. If the pressure be uniformly applied, and the ends expand, we get failure of elasticity when

$$Q > S_0.$$

Hence the apparent strength of a cylinder tested in this way would be about $\cdot 593$ of the strength of a cylinder tested under a distribution such as is usually assumed.

Further, if we consider the regions where the stress is greater than a given value S, we find that they consist of separate spaces, which join on to each other as S diminishes, the critical value for which this occurs being given by $S = 1{\cdot}01\,Q$. The regions of greatest stress consist therefore of a central core, which spreads out into a sort of hollow cone near the ends. If then we suppose fracture to occur over regions of greatest stress, we see why it is that the material breaks off in conical pieces at the ends.

Consider now the greatest strain theory. Let T_1, T_2, T_3 be the three principal stresses, and s_1, s_2, s_3 the corresponding stretches.

Then $$s_1 = \frac{1}{2\mu}\left(T_1 - \frac{\lambda}{3\lambda + 2\mu}(T_1 + T_2 + T_3)\right),$$

so that the greatest s will correspond to the greatest T, if T_1, T_2, T_3 have the same sign. This is our case everywhere, except in cases where $\widehat{\phi\phi} > 0$, and then $\widehat{\phi\phi}$ is so small that it still leaves the strain corresponding to $\widehat{ZZ}$ numerically the greatest.

We have then, remembering we have assumed $\lambda = \mu$, to investigate the values of

$$\widehat{ZZ} - \tfrac{1}{5}(\widehat{ZZ} + \widehat{RR} + \widehat{\phi\phi}) = 2\mu s_z.$$

This will be proportional to the greatest strain, except near $z = \pm c$, $r = 0$, where

$$\widehat{RR} - \tfrac{1}{5}(\widehat{ZZ} + \widehat{RR} + \widehat{\phi\phi}) = 2\mu s_r$$

should be taken. It is found, however, that at this point the strain is comparatively small, and the maximum strain there is a matter of indifference.

TABLE of s_z/s, where s = maximum stretch under the same uniform pressure.

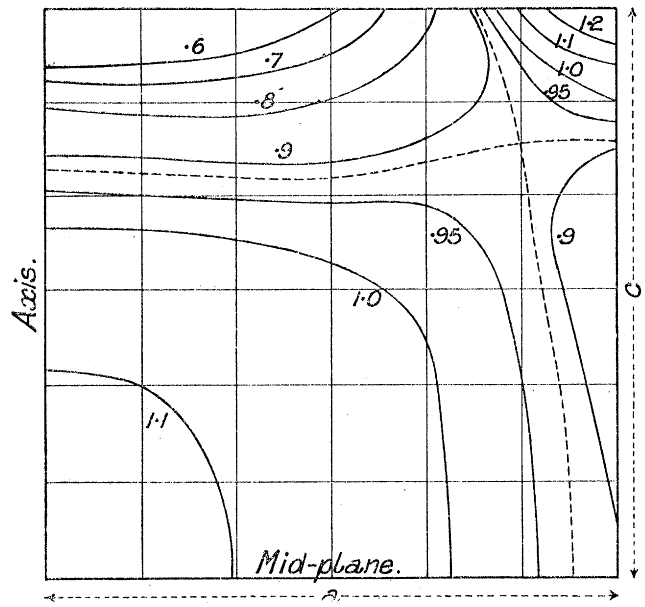
r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	1·13245	1·12729	1·10755	1·05938	·95338	·77296	·23743
$a/3$	1·10000	1·09589	1·07935	1·03756	·94751	·76590	·39034
$2a/3$	1·01133	1·01035	1·00509	·98813	·94446	·86416	·78311
a	·90246	·89563	·87724	·85716	·86591	·99395	1·57685

If we take therefore the "greatest stretch" theory, failure of elasticity still occurs at the perimeter of the ends, but this time only when the stress is $\frac{1}{1·577}$ (limiting stress in the case of uniformly compressed cylinder), so that although the apparent strength is less than in the uniform case, it is greater than if we adopt the "greatest stress" theory.

The lines of equal principal stretch s_z/s are shown in Diagram 13. They are drawn for only one quarter of the meridian plane, the rest being symmetrical. They present the same general characteristics as the curves of equal stress $\widehat{ZZ}$, with this difference, that the critical line corresponds to $s_z = ·915\,s$. Again, the caps or buttons at the ends are far larger; so that if pieces are cut out, they will be considerably larger than on the "greatest stress" theory. Also, looking at the inclination of the lines joining the corners to the critical points, *i.e.*, the points where the two branches of the critical line intersect, we see that the fragments, if approximately conical near their base, will probably be cut off at a much higher angle than in the previous case.

Let us now proceed to consider what happens if we adopt the third or "greatest stress-difference" theory of rupture. It is easy to see from the tables of $\widehat{RR}$, $\widehat{ZZ}$, and $\widehat{\phi\phi}$ that the greatest stress-difference is either $\widehat{RR}-\widehat{ZZ}$ or $\widehat{\phi\phi}-\widehat{ZZ}$. In the sixteen cases tabulated, for which $z > c/3$, the first of these is the greatest stress-difference, and in the twelve remaining cases the second is the greatest, although, as a matter of fact, the two stress-differences, for these twelve cases, do not diverge very much.

Diagram 13.—Distribution of Principal Stretch, s_z, inside the Cylinder (case of Compression between Rough Rigid Planes).



The number corresponding to each line = the value of s_z/s for that line.
-----------, critical line. s_z/s = ·915.

The actual greatest stress-difference is given in the following table :—

Table of (Maximum Stress-difference)/Q.

r.	$z = 0$.	$z = c/6$.	$z = 2c/6$.	$z = 3c/6$.	$z = 4c/6$.	$z = 5c/6$.	$z = c$.
0	1·13108	1·12022	1·08188	·99730	·82675	·55221	– ·21092
$a/3$	1·10270	1·09320	1·05892	·98861	·85786	·66217	·71287
$2a/3$	1·02486	1·01915	·99853	·97223	·93034	·91798	1·21101
a	·92695	·91821	·89246	·85845	·88177	1·04077	1·68635

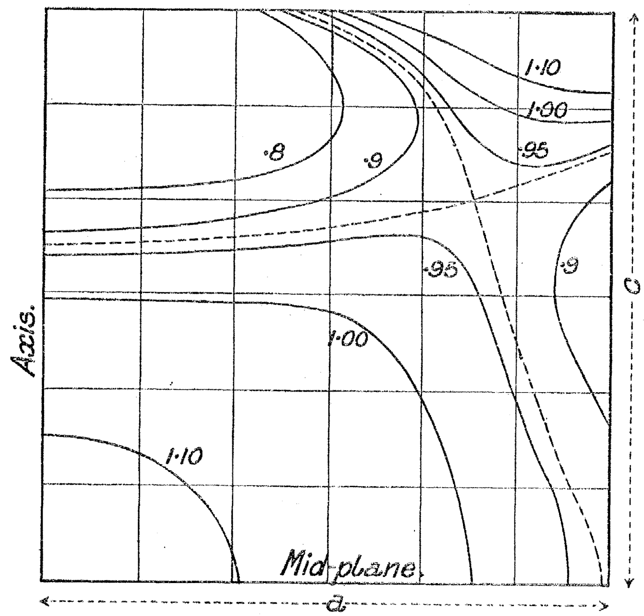
Here again plastic deformation will first occur round the perimeter of the ends when $Q = \left(\frac{1}{1·68635}\right)$ of the value it should have, on the same theory of rupture, in order to produce failure of elasticity in a uniformly compressed cylinder.

So far, then, this theory leads to the same results as the maximum stress theory.

Diagram 14 shows the distribution of maximum stress-difference. The lines of equal maximum stress-difference present very similar characteristics to those of equal maximum stretch. The critical line corresponds to a maximum stress-difference = ·933 Q.

Remarks similar to those used in the last case apply in this.

Diagram 14.—Distribution of Principal Stress-difference inside the Cylinder (case of Compression between Rough Rigid Planes).



The number corresponding to each line denotes the value of the ratio (maximum stress-difference : Q) for that line.

- - - - - - - - - - -, critical line. Stress-difference = ·933 Q.

§ 24. *Distorted Shape of the Curved Surface.*

If we work out the values of u when $r = a$, we find, after some reductions,

$$u_{r=a} = \mathrm{E}\alpha^5 \left\{ \frac{\zeta}{12} - \frac{1}{6}\frac{(z^2 - c^2)^2}{a^4} - \sum_1^\infty \frac{2r_n}{\alpha^3}(-1)^n \cos\frac{n\pi z}{c} \right\} . \quad . \quad . \quad . \quad (112),$$

where

$$r_n = \frac{\left\{(2\gamma + 1)\alpha + \frac{4}{\alpha}(8\gamma + 1)\right\} \mathrm{I}_1^2 - 4\mathrm{I}_0\mathrm{I}_1(2\gamma + 1)}{\gamma\alpha^2 \mathrm{I}_0^2 - (1 + \gamma\alpha^2)\mathrm{I}_1^2} \quad . \quad . \quad . \quad . \quad (113).$$

Now in the particular case we are dealing with, if α be large, $r_n(\alpha)$ approximates to

$$\frac{7}{2} - \frac{49}{4\alpha} + \frac{335}{16\alpha^2} + \frac{55}{4\alpha^3} + \ldots$$

Write then

$$r_n(\alpha) = \frac{7}{2} - \frac{49}{4\alpha} + \frac{335}{16\alpha^2} + \frac{55}{4\alpha^3} + r_n'(\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114),$$

and substitute in (112). We find, putting in for $\Sigma \frac{1}{n^2} \cos \frac{n\pi z}{c}$ and $\Sigma \frac{1}{n^4} \cos \frac{n\pi z}{c}$ their known values,

$$u_{r=a} = \mathrm{E}a^5 \left\{ \begin{array}{l} \dfrac{\zeta}{12} - \dfrac{11}{378}\dfrac{\pi^6}{729} + \dfrac{49}{180}\dfrac{\pi^4}{81} - \left(\dfrac{z^2 - c^2}{a^2}\right)^2 \left\{\dfrac{65}{96} + \dfrac{55}{2880}\left(\dfrac{z^2}{a^2} - \dfrac{3c^2}{a^2}\right)\right\} \\ - \dfrac{7c^3}{\pi^3 a^3}\sum_1^\infty \dfrac{(-1)^n}{n^3}\cos\dfrac{n\pi z}{c} - \dfrac{335}{8\pi^5}\dfrac{c^5}{a^5}\sum_1^\infty \dfrac{(-1)^n}{n^5}\cos\dfrac{n\pi z}{c} \\ - 2\sum_1^\infty (-1)^n \dfrac{r_n'}{a^3}\cos\dfrac{n\pi z}{c} \end{array} \right\} \quad . \quad (115),$$

where, putting in now $\pi a = 3c$, we have the following values for r_n' :—

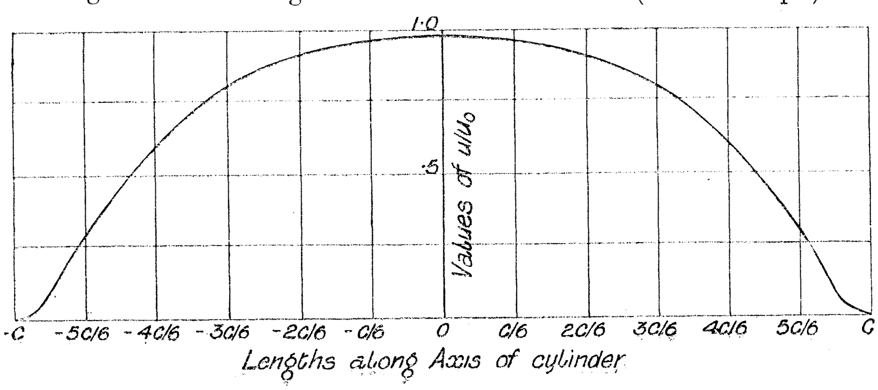
n.	$r_n'(a)$.
1 . . .	$-$·42430
2 . . .	$-$·01807
3 . . .	$-$·00223
4 . . .	$-$·00063
5 . . .	$-$·00024
6 . . .	$-$·00010

Whence, using the methods of § 20, we find the following values for u/u_0, where u_0 = lateral expansion of a cylinder of the same dimensions under the same total pressure uniformly distributed :—

z.	u/u_0.
0	·97861
$c/6$	·96325
$2c/6$	·91118
$3c/6$	·80402
$4c/6$	·61209
$5c/6$	·30863
c	·00000

These results are exhibited in Diagram 15. We see that the cylinder has a single bulge in the centre, and indeed it is easy to verify that $(d^2u/dz^2)_0 < 0$ by differentiating equation (115) and putting in the numerical values.

Diagram 15.—Showing Distortion of Curved Surface (second example).



This agrees with the figures shown by BACH in his 'Elasticität und Festigkeit' (figs. 2 and 3, Par. 11) of the distorted shapes of such cylinders. There are, however, in the possession of Professor KARL PEARSON, at University College, London, specimens of iron which have been strongly compressed, so that the strain has been large and permanent and the meridian section of the distorted curved surface is an obviously wavy curve, with *two* maxima, one on either side of the mid-section. With regard to the apparent disagreement here between theory and practice, I would observe that these specimens have been subjected to enormous stresses, for which the equations of elasticity certainly do not apply, and probably are not even an approximation; in the second place, the specimens I have seen are longer, compared with their diameter, than the cylinder of the present numerical example, so that it is easy to see why the conclusions above need not apply to these specimens.

§ 25. *Apparent Young's Modulus and Poisson's Ratio.*

We find that the total shortening of the bar

$$= -2(w)_{z=c} = 2\left[\mathrm{Q}/\mu \frac{\gamma\left\{\frac{8}{45}c^5 - \frac{ca^4\zeta}{6}\right\} - \frac{ca^4}{12}(1-\gamma)}{(4\gamma-1)\left\{\frac{8}{45}c^4 - \frac{a^4\zeta}{6}\right\} - \frac{a^4}{12}}\right] \quad . \quad . \quad . \quad . \quad (116),$$

where

$$\zeta = 24\sum_1^\infty \frac{1}{\alpha^3}\frac{\left[(2\gamma+1)\alpha + \frac{4}{\alpha}(8\gamma+1)\right] - \frac{4\mathrm{I}_0}{\mathrm{I}_1}(2\gamma+1)}{\left(\frac{\mathrm{I}_0}{{}_1\mathrm{I}}\right)^2\gamma\alpha^2 - (1+\gamma\alpha^2)}.$$

Hence the apparent YOUNG's modulus $\mathrm{E_Y}' = \mathrm{Q}c/(w)_{z=c}$,

$$\mathrm{E_Y}' = \mu\frac{\left[(4\gamma-1)\left\{\frac{8}{45}c^4 - \frac{a^4\zeta}{6}\right\} - \frac{a^4}{12}\right]}{\gamma\left\{\frac{8}{45} \quad - \frac{a^4\zeta}{6}\right\} - \frac{a^4}{12}(1-\gamma)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117).$$

Now if α be small, *i.e.*, if the bar be very long

$$\mathrm{I_0/I_1} = \frac{2}{\alpha}\left(1 + \frac{\alpha^2}{8} - \frac{7\alpha^4}{192} + \ldots\right)$$

and

$$\zeta = 96\,\Sigma\,\frac{1}{\alpha^4}\left(1 + \frac{36\gamma+7}{4\gamma-1}\,\frac{\alpha^4}{96}\right),$$

retaining only terms of 4th order,

$$= \frac{96}{90}\frac{c^4}{a^4} + \nu\frac{36\gamma+7}{4\gamma-1},$$

where ν is the greatest integral value for which $\alpha = \nu\pi a/c$ makes the above approximation sufficient.

Hence if a/c is very small, ν may be large, and thus although the first term in $a^4\zeta/6$ cancels $\frac{8}{45}c^4$, yet the terms in curled brackets will become indefinitely large compared with the other terms. Thus, for a very long cylinder,

$$E_Y' = \frac{\mu(4\gamma - 1)}{\gamma} = E_Y,$$

where E_Y is the true YOUNG's modulus.

On the other hand, for a very short cylinder, c/a is very small, and ζ being of the order $\frac{24}{\pi^3}\frac{c^3}{a^3} \times \frac{2\gamma+1}{\gamma}\sum_1^\infty \frac{1}{n^3}$, the two leading terms in the numerator and denominator of the right-hand side of (117) are negligible and

$$E_Y' = \frac{\mu}{1-\gamma} = \frac{\gamma}{(1-\gamma)(4\gamma-1)} E_Y.$$

This is identical with the modulus of compression for a cylinder which is prevented from expanding laterally by a constant pressure applied to the sides. So that we see that for a very flat disc the effect on the modulus of compression is the same, whether the lateral expansion be prevented by means of shearing-stress over the flat ends, or by hydrostatic pressure over the curved surface.

The apparent YOUNG's modulus for intermediate cases will be between those two values (these, for uniconstant isotropy, being E_Y and $6/5\,E_Y$). Thus, in the given example, where $\gamma = 2/3$,

$$E_Y' = 1\cdot0498 E_Y.$$

POISSON's ratio comes out to be apparently ·2690 instead of ·2500.

Thus the errors in the values of YOUNG's modulus and POISSON's ratio, as deduced from an experiment with cylinders under the given conditions, will be 5 per cent. and 7·6 per cent. respectively.

§ 26. *Solution involving Discontinuities at the Perimeter of the Plane Ends.*

In § 13 it was stated that a solution, obtained by methods strictly analogous to those used in that section, but which neglected the condition that the shear $\widehat{rz}$ should be continuous at the perimeter of the plane ends, could be found.

It seems of interest to give, for purposes of comparison, the expressions for the displacements and stresses, deduced from this solution. They are :

$$\left.\begin{aligned} u &= u_0 r + u_1 r^3/3 + D r z^2/2 + \sum_1^\infty \left\{ -\frac{A_1}{k} I_1(kr) - \frac{C}{k} r I_0(kr) \right\} \cos kz \\ w &= w_0 z + w_1 z^3/3 + \sum_1^\infty \left\{ \frac{A_2}{k} I_0(kr) + \frac{C}{k} r I_1(kr) \right\} \sin kz \end{aligned}\right\} . \quad (118)$$

where

$$\left.\begin{aligned} u_1 &= -\tfrac{3}{8}(1-\gamma)\,D \\ w_1 &= -\gamma D \end{aligned}\right\} \quad (119),$$

$$\left.\begin{aligned} A_2 - A_1 &= 2C/k\gamma \\ A_2 + A_1 &= -\frac{2C}{k}\frac{\alpha I_0(\alpha)}{I_1(\alpha)} + \frac{2Dac}{n\pi}(-1)^n \end{aligned}\right\} \quad (120),$$

where

$$k = n\pi/c.$$

Also

$$C = -\frac{D}{k}(-1)^n \frac{\gamma\alpha I_1(\alpha)(4\gamma+1) - \gamma\alpha^2 I_0(\alpha)}{\gamma\alpha^2 (I_0^2(\alpha) - I_1^2(\alpha)) - I_1^2(\alpha)} \quad (121),$$

and

$$\left.\begin{aligned} u_0 &= D\left[\tfrac{1}{8}(1-\gamma)a^2 - \frac{c^2\zeta}{3}\right] \\ w_0 &= D\left(\frac{1-\gamma}{2\gamma-1}\right)\left[\tfrac{1}{4}a^2 + \frac{2\gamma c^2}{3(1-\gamma)}(\zeta+\gamma-1)\right] \\ \text{where}\quad D &= -\frac{Q}{\mu}\,\frac{2\gamma-1}{\frac{a^2}{4} - \frac{2}{3}c^2[\zeta(4\gamma-1)+(3\gamma-1)]} \\ \text{and}\quad \zeta &= -\frac{3}{\pi^2}\sum_1^\infty \frac{1}{n^2}\left\{\frac{I_1^2(4\gamma+1) - \alpha I_0 I_1}{\gamma\alpha^2(I_0^2 - I_1^2) - I_1^2}\right\} \end{aligned}\right\} \quad (122).$$

The method by which the constants are obtained is precisely the same as that used in §§ 14, 15.

The expressions for the stresses are given below:

$$\frac{\widehat{rr}}{\mu} = D\left[\tfrac{1}{4}(1+\gamma)(a^2-r^2) + 2\gamma\left(z^2 - \frac{c^2}{3}\right) + \frac{2ac}{\pi}\sum_1^\infty \frac{(-1)^n}{n}\left[\frac{I_1(\rho)}{\rho I_1(\alpha)} - \frac{I_0(\rho)}{I_1(\alpha)}\right]\cos\frac{n\pi z}{c}\right]$$
$$+ \sum_1^\infty \frac{2C}{k}\left[\left\{1 + \frac{\alpha I_0}{I_1}\right\} I_0(\rho) - \left\{\frac{\alpha I_0}{I_1} + \frac{1}{\gamma}\right\}\frac{I_1(\rho)}{\rho} - \rho I_1(\rho)\right]\cos\frac{n\pi z}{c} \quad (123),$$

$$\frac{\widehat{\phi\phi}}{\mu} = D\left[\tfrac{1}{4}(1+\gamma)a^2 + 2\gamma\left(z^2 - \frac{c^2}{3}\right) - \tfrac{1}{4}(3\gamma-1)r^2 - \frac{2ac}{\pi}\sum_1^\infty \frac{(-1)^n}{n}\frac{I_1(\rho)}{\rho I_1(\alpha)}\cos\frac{n\pi z}{c}\right]$$
$$+ \sum_1^\infty \frac{2C}{k}\left[\left(1 - \frac{1}{\gamma}\right) I_0(\rho) + \left(\frac{1}{\gamma} + \frac{\alpha I_0}{I_1}\right)\frac{I_1(\rho)}{\rho}\right]\cos\frac{n\pi z}{c} \quad (124),$$

$$\frac{\widehat{zz} + Q}{\mu} = D\left[\frac{2c^2}{3} + \frac{\widehat{2\gamma-1}a^2}{4} - z^2 - \frac{\widehat{2\gamma-1}}{2}r^2 + \frac{2ac}{\pi}\sum_1^\infty \frac{(-1)^n}{n}\frac{I_0(\rho)}{I_1(\alpha)}\cos\frac{n\pi z}{c}\right]$$
$$+ \sum_1^\infty \frac{2C}{k}\left[\left\{2 - \frac{\alpha I_0}{I_1}\right\} I_0(\rho) + \rho I_1(\rho)\right]\cos\frac{n\pi z}{c} \quad (125),$$

$$\frac{\widehat{rz}}{\mu} = D\left[rz + \frac{2ac}{\alpha}\sum_1^\infty \frac{(-1)^n}{n}\frac{I_1(\rho)}{I_1(\alpha)}\sin\frac{n\pi z}{c}\right]$$
$$+ \sum_1^\infty \frac{2C}{k}\left(\rho I_0(\rho) - I_1(\rho)\frac{\alpha I_0}{I_1}\right)\sin\frac{n\pi z}{c} \quad (126).$$

It is now easy to see, if we bear in mind that when n and therefore α is large, CI_0 remains finite, as appears on examination of (121), how it is that the stresses at the boundary become infinite.

For in both $\widehat{zz} + Q$ and $\widehat{\phi\phi}$ we have, when $r = a$, terms of order $1/n$, when n is large. These are of alternate sign, C containing $(-1)^n$. But if $z = \pm c$ they become all of the same sign, and the series become logarithmically infinite.

§ 27. *Summary of Results.*

Looking back upon the results obtained, we notice :

(*a.*) That the three solutions we have been considering successively are only the simplest of an infinite series of solutions, which are continually growing more complicated ; for we need not necessarily stop, as has been done, at terms of the fifth degree, but might go on to terms of any degree in r and z and thus construct, as it were, solutions of successive orders. We should then have an infinite number of free constants, which might be determined by introducing further limitations at the plane ends, such as, for instance, restricting u to be zero at every point and not merely along the perimeter.

The analytical complexity of such a complete solution would, however, be very great, and would render it quite beyond the reach of arithmetical expression, and consequently valueless for the purposes of the engineer and the physicist. No attempt has therefore been made to develop this solution, although, as an analytical possibility, it appears interesting.

(*b.*) That the different solutions all agree in giving the perimeter of the plane ends as the locus of the points where the elastic limit will first be passed, one of these solutions actually making the stress infinite at this perimeter.

In the more important solution, however, where continuity and finiteness are preserved, the conclusion still holds, and, further, is independent of whatever theory of tendency to rupture we adopt, whether we suppose it due to maximum stress, to maximum stretch or squeeze, or to maximum shear or stress-difference.

(*c.*) That in the numerical example considered, plastic deformation begins to occur round the perimeter for a stress between 2/3 and 1/2 of that which is required to cause a cylinder under uniform pressure to pass the elastic limit.

This is apparently in contradiction with the results of engineering experience, both UNWIN and PERRY stating that blocks of stone or cement, pressed between millboard, which hinders the expansion of the ends, show greater strength than the same blocks when the ends are allowed to expand.

The key to this appears to be found in a remark of UNWIN, which Professor EWING confirms, that the lead sheets do not merely *allow* the expansion of the block, they *force* it, *i.e.*, lead in its plastic state will expand more than the stone or cement would do laterally under a uniform axial pressure.

But the solution, when the ends are compelled to expand by a given quantity α_0, is easily deducible from that given for non-expanding ends. Thus, let u_1, w_1 be the values of u and w in the case worked out, when $Q = 1$, and let us write

$$u = P\left(\frac{\lambda r}{2\mu(3\lambda + 2\mu)}\right) + Ru_1,$$

$$w = P\left(\frac{-(\lambda + \mu)z}{\mu(3\lambda + 2\mu)}\right) + Rw_1.$$

Then this will satisfy all the conditions, provided

$$\frac{P}{\mu}\frac{\lambda}{6\lambda + 4\mu} \times a = \alpha_0$$

and

$$P + R = Q.$$

Therefore

$$u = \frac{\alpha_0 r}{a} + \left(Q - \frac{\alpha_0}{a}\frac{\mu}{\lambda}(6\lambda + 4\mu)\right)u_1,$$

$$w = \frac{-2(\lambda + \mu)}{\lambda}\frac{\alpha_0 z}{a} + \left(Q - \frac{\alpha_0}{a}\frac{\mu}{\lambda}(6\lambda + 4\mu)\right)w_1,$$

giving the solution under a given mean pressure Q, which produces a flow α_0 of a lead plate, and thereby constrains the ends of the test piece to expand by that amount.

The principal stress at the perimeter of the section is now

$$P + (1{\cdot}686)R = 1{\cdot}686\,Q - {\cdot}686\,P,$$

and if P, *i.e.*, α_0, be made large enough, this can be made much smaller than Q. It begins to be smaller than Q as soon as the expansion induced by the flow of the lead is greater than the natural expansion of the stone under uniform pressure.

On the other hand, the principal stress at the centre of the plane ends is

$$P + {\cdot}686\,R = {\cdot}686\,Q + {\cdot}314\,P,$$

and this again may be made great by making α_0 large.

The principal stress-differences are :

at the perimeter	$1{\cdot}686\,Q - {\cdot}686\,P$
at the centre	$P - {\cdot}211\,R = -{\cdot}211\,Q + 1{\cdot}211\,P.$

Hence we see that whatever theory of failure we adopt, if the ends are forced to expand, so that $P > Q$, the material first becomes plastic (or else breaks) at the centre of the cross-section, the strength of the test-piece diminishing as P increases, but having no definite value. That some such thing as this does really occur in practice is very well shown by the results published by UNWIN ('The Testing of Materials of

Construction'), where blocks show less strength when three sheets of lead are introduced between the compressing planes and the test piece than when one sheet only is introduced, the lateral flow being greater in the first case owing to the larger amount of lead.

It would seem, therefore, as if the true strength of a cylinder were really greater than its strength as tested either between millboards or between lead sheets, and not, as Professor PERRY states in his 'Applied Mechanics,' equal to the strength shown in the lead test—this test, as we see, leading to results that are not definite, but vary with the expansion of the lead. The millboard test, however, which is advocated by UNWIN, should give a constant value, although it is not the value which would hold for a cylinder under uniform pressure.

(*d.*) Diagrams 12–14 suggest an explanation of the fact that, when short cylinders are strongly compressed between very hard surfaces, pieces are sometimes cut out at the ends of an approximately conical shape. The same occurs when spherical pieces of metal, such as ball-bearings, are compressed between parallel plates. This is usually explained by saying that the material breaks along the planes of principal shear. On the other hand, it may be argued simply that rupture should take place over the regions of greatest stress. These are near the perimeter at the ends, and gradually close in upon the centre, forming hollow caps.

Further, in the case of the lead tests, where $P > Q$, this state of things is reversed, and the material should give way from the inside, so that we should expect it to split axially, and possibly along meridian planes as well. That this is what really occurs can be verified by referring to the figures in the chapter on testing of stone in UNWIN'S 'Testing of Materials of Construction.'

(*e.*) The results both of this and of the first problem show us how unreliable any experiments on short cylinders must be, which have in view the determination, by tensile strain, of either YOUNG'S modulus or POISSON'S ratio. Thus any results obtained in such a case without the dimensions and the mode of application of the stress being exactly specified, would not justify us in general in drawing any conclusions as to whether a given material possesses or not uniconstant isotropy.

§ 28. *The Third Problem. Case of Torsion. Expressions for the Displacement and Stresses.*

I now proceed to consider a case where u and w are zero, that is, where we have to deal with the solution in v, which we have seen is independent of the others.

We have in the notation of § 3

$$(\vartheta^2 + D^2)\, v = 0\,.$$

Hence, excluding K-functions, since the solution must be finite and continuous at the origin, we have

$$v = \Sigma\,(A_n \sin kz + B_n \cos kz)\, I_1(kr)\,.$$

Now, if the torsion be symmetrical on either side of the mid-section, we have $v=0$, when $z=0$, therefore

$$B_n = 0.$$

But also

$$\widehat{\phi z} = \mu \frac{dv}{dz} = 0 \text{ at the ends.}$$

Therefore

$$\cos kc = 0 \text{ or } kc = \overline{2n+1}\pi/2.$$

Also

$$\widehat{r\phi} = \mu r \frac{d}{dr}\left(\frac{v}{r}\right) = \Sigma \mu A_n \sin kz \,.\, r \frac{d}{dr} \frac{I_1(kr)}{r}$$

$$= \Sigma \mu A_n \sin kz \,.\, k I_2(kr).$$

By the well-known property of the I-functions

$$\frac{d}{dx}\left(\frac{I_n(x)}{x^n}\right) = \frac{I_{n+1}(x)}{x^n}.$$

Now suppose that the cylinder is subjected to a certain system of transverse surface shears, so that $\widehat{r\phi}$ can be expanded in a FOURIER'S series in the form

$$(\widehat{r\phi})_a = \sum_0^\infty c_n \sin \frac{\overline{2n+1}\pi z}{2c}.$$

Then

$$\mu k A_n I_2(\alpha) = c_n$$

or

$$A_n = \frac{c_n}{\mu k} \frac{1}{I_2(\alpha)}.$$

Hence

$$v = \sum_0^\infty \frac{c_n}{\mu k} \frac{I_1(\rho)}{I_2(\alpha)} \sin \frac{\overline{2n+1}\pi z}{2c} \quad \ldots \ldots \quad (127),$$

$$\widehat{r\phi} = \sum_0^\infty c_n \frac{I_2(\rho)}{I_2(\alpha)} \sin \frac{\overline{2n+1}\pi z}{2c} \quad \ldots \ldots \quad (128),$$

$$\widehat{\phi z} = \sum_0^\infty c_n \frac{I_1(\rho)}{I_2(\alpha)} \cos \frac{\overline{2n+1}\pi z}{2c} \quad \ldots \ldots \quad (129),$$

where ρ α have the same meanings as before, viz., kr, ka.

In the case where a/c is very small, or the cylinder is very long relatively to its diameter, we may obtain a first approximation by retaining only the first terms in the expression for the I's. Proceeding as in § 6, we find

$$\left.\begin{aligned} v &= \frac{4r}{\mu a^2}\sum_0^\infty \frac{c_n}{k^2}\sin kz \\ \widehat{r\phi} &= \frac{r^2}{a^2}\sum_0^\infty c_n \sin kz \\ \widehat{\phi z} &= \frac{4r}{a^2}\sum_0^\infty \frac{c_n}{k}\cos kz \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (130).$$

Now if $(\widehat{r\phi})_a = \psi(z)$,

we see that

$$\widehat{r\phi} = r^2\psi(z)/a^2,$$

$$\widehat{\phi z} = (4r/a)\int_z^c \psi(z)\,dz,$$

$$v = \frac{4r}{\mu a^2}\int_0^z dz\int_z^c \psi(z)\,dz.$$

Now if M be the total torsion moment up to any cross-section,

$$\mathrm{M} = 2\pi a^2\int_z^c \psi(z)\,dz,$$

$$\widehat{\phi z} = \frac{2r}{\pi a^4}\mathrm{M},$$

v/r = angle turned through by a radius = θ say.

Therefore $$\theta = \frac{2}{\pi\mu a^4}\int_0^z \mathrm{M}dz,$$

$$\frac{d\theta}{dz} = \text{torsion at the point} = \tau.$$

Therefore $$\tau = \frac{2\mathrm{M}}{\pi\mu a^4}.$$

Therefore $$\mathrm{M} = \mu \times \frac{\pi a^4}{2} \times \tau \text{ and } \widehat{\phi z} = \mu\tau r.$$

But these are the actual formulæ connecting the torsion with the applied couple and with the shear across the section for a circular cylinder.

We see, then, that the usual formulæ continue to hold, to the first approximation, when the forces applied to the surface of the cylinder vary with z, provided we define our torsion-couple at any section (much as the bending moment at any section of a beam is defined), as the couple of all the external applied forces to one side of that cross-section.

It is interesting to note also that, to this approximation, there is no distortion of the cross-sections, v/r being constant for the section. Straight radii therefore remain straight radii.

Further, $\widehat{r\phi} = (r^2/a^2)$ (its value at the boundary).

In other words, the transverse shearing-stress across cylinders coaxial with the given one is zero for sections where there is no such external applied stress, and for other sections diminishes rapidly along the radius as we go inwards, so that near the centre it is always small compared with $\widehat{\phi z}$.

There is one very important point to be noted with regard to this method of approximation: ρ and α increase with n, and therefore, however small a/c may be, so long as it remains finite, we still reach a value of n, for which it is not justifiable to take for I_1 and I_2 the first terms of their expansions in positive integral powers of the argument. If, however, we stop at the νth term, where ν is finite, then if R_ν is the remainder after ν terms of the series

$$\sum_0^\infty c_n \frac{I_1(\rho)}{I_2(\alpha)} \cos \frac{\overline{2n+1}\pi z}{2c}, \text{ for example,}$$

and if, on the other hand, the numerical value of the difference $\frac{4r}{ka^2} - \frac{I_1(\rho)}{I_2(\alpha)} < \epsilon$ for all values of n not greater than ν, where ϵ is a quantity which depends upon c/a, and which can be made as small as we please by making c/a small enough, then the difference

$$\sum_0^\infty c_n \frac{I_1(\rho)}{I_2(\alpha)} \cos \frac{\overline{2n+1}\pi z}{2c} - \sum_0^\infty \frac{c_n}{k}\frac{4r}{a^2} \cos \frac{\overline{2n+1}\pi z}{2c}$$

must be numerically less than

$$|R_\nu| + |R_\nu'| + \nu\epsilon,$$

where $|x|$ denotes the numerical value or modulus of x, and R_ν' is the remainder after ν terms of the series

$$\sum_0^\infty \frac{c_n}{k}\frac{4r}{a^2} \cos \frac{\overline{2n+1}\,\pi z}{2c}.$$

Now if both the original series and the approximate series are uniformly convergent, then by giving ν a certain value, $|R_\nu|$ and $|R_\nu'|$ can both be made less than a certain small quantity $\eta/3$ which tends to zero when ν tends to infinity, and that for all values of z.

Now make c/a so small that $\epsilon < \eta/3\nu$, which we can always do so long as ν is finite. Then the difference between the two series is numerically $< \eta$, and the approximation holds.

If, however, for any value of z, it becomes impossible to assign an upper limit to R_ν or R_ν', *i.e.*, if either series cease to be uniformly convergent, then we should have to increase ν indefinitely in order to make $|R_\nu| < \eta/3$, and therefore to modify ϵ, so that no limiting value of c/a (which should, of course, be independent of z) could be found, and the approximation need not necessarily hold. As a matter of fact, it is shown in § 29 to fail for particular cases. This is true *à fortiori*, if either series cease to be convergent at all.

The same remarks apply in their entirety to the process of approximation given in § 6, and further, to the approximate expressions given by Professor POCHHAMMER in his investigation on the bending of beams ('Crelle,' vol. 81).

§ 29. *Special Case of Two Discontinuous Rings of Shear.*

Suppose that we have the following system of values for $\widehat{\phi z}$:—

$$\widehat{\phi z} = \text{T if } c - e < z < c,$$
$$\widehat{\phi z} = 0 \text{ if } -c + e < z < c - e,$$
$$\widehat{\phi z} = -\text{T if } -c < z < -c + e,$$

so that we have a cylinder twisted by two equal and opposite rings of transverse shear extending over lengths e of the cylinder, near the ends. Then we find easily

$$c_n = \frac{4\text{T}}{(2n+1)\pi}(-1)^n \sin\frac{(2n+1)\pi e}{2c}$$

with the following values of the displacements and stresses:

$$\left.\begin{aligned} v &= \sum_0^\infty \frac{8\text{T}c}{\mu(2n+1)^2\pi^2}(-1)^n \frac{\text{I}_1(\rho)}{\text{I}_2(\alpha)} \sin\frac{\widehat{2n+1}\,\pi e}{2c} \sin\frac{\widehat{2n+1}\,\pi z}{2c} \\ \widehat{r\phi} &= \sum_0^\infty \frac{4\text{T}}{(2n+1)\pi}(-1)^n \frac{\text{I}_2(\rho)}{\text{I}_2(\alpha)} \sin\frac{\widehat{2n+1}\,\pi e}{2c} \sin\frac{\widehat{2n+1}\,\pi z}{2c} \\ \widehat{\phi z} &= \sum_0^\infty \frac{4\text{T}}{(2n+1)\pi}(-1)^n \frac{\text{I}_1(\rho)}{\text{I}_2(\alpha)} \sin\frac{\widehat{2n+1}\,\pi e}{2c} \cos\frac{\widehat{2n+1}\,\pi z}{2c} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (131).$$

Now it is easy to see that in this case the conditions for uniform convergency are satisfied, except at the boundary, and except with regard to the stress $\widehat{r\phi}$, whose approximate expression is not uniformly convergent, being in fact discontinuous for $z = \pm(c - e)$.

At the boundary, $\text{I}_1(\alpha)/\text{I}_2(\alpha)$ tends to unity with n, its approximate expression, when α is large, being

$$\frac{\text{I}_1(\alpha)}{\text{I}_2(\alpha)} = 1 + \frac{3}{2\alpha} + \frac{15}{8\alpha^2} + \frac{15}{8\alpha^3} + \ldots \quad . \quad . \quad . \quad (132).$$

Hence v is always uniformly convergent and its approximate expression likewise, so for it the approximation, for sufficiently small values of c/a, holds throughout.

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For $\widehat{r\phi}$ the series is non-uniformly convergent for $r = a$ in the neighbourhood of the sections $z = \pm (c - e)$ owing to the series having a finite discontinuity. For $\widehat{\phi z}$ the approximation certainly fails, for $r = a$, in the neighbourhood of $z = \pm (c - e)$ for part of the expression for $\widehat{\phi z}$ is the series $\sum_0^\infty 1/(2n + 1)$, which is divergent.

Hence, if we are in such a case to use our approximations for the stresses, we must exclude the sections where the applied stress is discontinuous and their immediate neighbourhood from consideration.

It will be found that similar remarks apply to the example of pull given in § 7, and also to the example given by Professor POCHHAMMER in his paper on bending (*loc. cit.*), in which he also deals with discontinuous systems of stress, so that his approximate expressions leave us in the dark as to what does really happen at points of support, the cross-sections in the neighbourhood of such points being, for reasons analogous to those developed above, excluded from the regions where his approximations hold.

Before proceeding to an actual numerical concrete case, we may notice that $\widehat{\phi z}$ becomes infinite at the points $z = \pm (c - e)$ for the causes stated above. Hence any discontinuity in a system of transverse shears applied to the surface of a cylinder, or any such shear transmitted by a grip applied to a portion of the material projecting at a sharp angle, will produce in the neighbourhood a very great stress across the section. It would seem, therefore, that a cylinder treated in this way would be likely to experience plastic flow, or to rupture, not in the middle, but near the points where it is seized.

§ 30. *Approximations on the Boundary when the Cylinder is short.*

When the cylinder is short, so that α becomes rapidly large, we may use the method employed in §§ 8, 9, and 19, availing ourselves of the approximation (132). We then find :—

$$v = \frac{4\mathrm{T}c}{\mu\pi^2}\left\{\frac{\pi}{4}\left(1 - \frac{z-e}{c}\right)\log_\epsilon \left| \cot \left\{\frac{\pi}{4}\left(1 - \frac{z-e}{c}\right)\right\}\right| - \frac{\pi}{4}\left(1 - \frac{z+e}{c}\right)\log_\epsilon \left| \cot\left[\frac{\pi}{4}\left(1 - \frac{z+e}{c}\right)\right]\right|\right.$$

$$+ \frac{4\mathrm{T}c}{\mu\pi^2}\left\{\tfrac{1}{2}\int_0^{\frac{\pi}{2}\left(1-\frac{z-e}{c}\right)} x \operatorname{cosec} x\, dx - \tfrac{1}{2}\int_0^{\frac{\pi}{2}\left(1-\frac{z+e}{c}\right)} x \operatorname{cosec} x\, dx\right\}$$

$$+ \tfrac{3}{4}\frac{\mathrm{T}\pi}{\mu c}\left[\begin{array}{l} ez \text{ (from } z = 0 \text{ to } z = c - e) \\ \text{and} \\ ez - \tfrac{1}{2}(z - c + e)^2 \text{ (from } z = c - e \text{ to } z - c) \end{array}\right]$$

$$+ 15 \frac{\mathrm{T}c}{\mu\pi^2} \sum_0^\infty (-1)^n \left(\frac{1}{(2n+1)^4} + \frac{1}{(2n+1)^5} \right) \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi z}{2c}$$

$$+ \frac{8\mathrm{T}c}{\mu\pi^2} \sum_0^\infty \frac{(-1)^n}{(2n+1)^2} \left\{ \frac{\mathrm{I}_1(\alpha)}{\mathrm{I}_2(\alpha)} - 1 - \frac{3}{2\alpha} - \frac{15}{8\alpha^2} - \frac{15}{8\alpha^3} \right\} \sin \frac{\widehat{2n+1}\pi e}{2c} \sin \frac{\widehat{2n+1}\pi z}{2c} \quad . \quad . \quad (133),$$

$$\widehat{\phi z} = \frac{\mathrm{T}}{\pi} \log_e \left| \frac{\tan\left(\frac{\pi}{4} - \frac{\pi}{4}\frac{z-e}{c}\right)}{\tan\left(\frac{\pi}{4} - \frac{\pi}{4}\frac{z+e}{c}\right)} \right| + \tfrac{3}{4}\frac{\pi \mathrm{T}}{c} \left[\begin{array}{l} e \text{ (from } z = 0 \text{ to } z = c - e) \\ \text{and} \\ c - z \text{ (from } z = c - e \text{ to } z = c) \end{array} \right]$$

$$+ \tfrac{15}{2} \frac{\mathrm{T}}{\pi} \sum_0^\infty (-1^n \left(\frac{1}{(2n+1)^3} + \frac{1}{(2n+1)^4} \right) \sin \frac{(2n+1)\pi e}{2c} \cos \frac{\widehat{2n+1}\pi z}{2c}$$

$$+ \frac{4\mathrm{T}}{\pi} \sum_0^\infty \frac{(-1)^n}{(2n+1)} \left(\frac{\mathrm{I}_1(\alpha)}{\mathrm{I}_2(\alpha)} - 1 - \frac{3}{2\alpha} - \frac{15}{8\alpha^2} - \frac{15}{8\alpha^3} \right) \sin \frac{\widehat{2n+1}\pi e}{2c} \cos \frac{\widehat{2n+1}\pi z}{2c} \quad (134),$$

where in the last Σ in both equations only a comparatively small number of terms need be taken.

§ 31. *Numerical Example. Values of the Coefficients and of the Displacement and Stresses.*

Take a cylinder such that $\pi a/2c = 1$, and let $e = c/2$, so that the distribution of stress is as shown in fig. 4. Then $\alpha = 2n + 1$, and we find :

$$v = \frac{4\sqrt{2}\mathrm{T}c}{\mu\pi^2} (v_0 \sin \pi z/2c - v_1 \sin 3\pi z/2c - v_2 \sin 5\pi z/2c + v_3 \sin 7\pi z/2c + v_4 \sin 9\pi z/2c$$
$$- v_5 \sin 11\pi z/2c - \ldots)$$

$$\widehat{r\phi} = \frac{2\sqrt{2}\mathrm{T}}{\pi} (t_0 \sin \pi z/2c - t_1 \sin 3\pi z/2c - \ldots)$$

$$\widehat{\phi z} = \frac{2\sqrt{2}\mathrm{T}}{\pi} (s_0 \cos \pi z/2c - s_1 \cos 3\pi z/2c - \ldots)$$

the law of the signs being obvious, and the v's, t's, and s's being given below :

TABLE of t_n and s_n.

n.	t_n.				s_n.			
	$r = \cdot 2a$.	$r = \cdot 4a$.	$r = \cdot 6a$.	$r = a$.	$r = \cdot 2a$.	$r = \cdot 4a$.	$r = \cdot 6a$.	$r = a$.
0	·036956	·149306	·341554	1·000000	·740348	1·502982	2·310929	4·163295
1	·006884	·030078	·078098	·333333	·046574	·106104	·195552	·586933
2	·001551	·007871	·025651	·200000	·006457	·018173	·045167	·278032
3	·000331	·002073	·009064	·142857	·001021	·003803	·013485	·179752
4	·000068	·000547	·003339	·111111	·000169	·000813	·004522	·132502
5	·000013	·000145	·001266	·090909	·000029	·000212	·001619	·104848
6	·000003	·000039	·000492	·076923	·000005	·000053	·000604	·086720
7	·000001	·000011	·000195	·066667	·000001	·000014	·000232	·073927
8	·000000	·000003	·000078	·058824	·000000	·000004	·000091	·064419
9	·000000	·000001	·000032	·052632	·000000	·000001	·000036	·057075

Fig. 4.

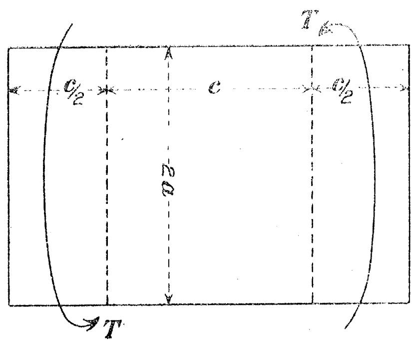


TABLE of v_n.

$r = \cdot 2a$.	$r = \cdot 4a$.	$r = \cdot 6a$.	$r = a$.
·740348	1·502982	2·310929	4·163295
·015525	·035368	·065184	·195644
·001291	·003635	·009033	·055606
·000146	·000543	·001926	·025679
·000019	·000097	·000502	·014722
·000003	·000019	·000147	·009532
·000000	·000004	·000046	·006671
·000000	·000001	·000015	·004928
·000000	·000000	·000005	·003789
·000000	·000000	·000002	·003004

From these, using the formulæ of approximation given in the last section, when $r = a$, we obtain the values of v and the two stresses. I have tabulated them in the form v/v_0 and (stress) $/\mathrm{T}_0$ where v_0, T_0 are the greatest values of the displacement and of the shear respectively in a cylinder of the same length, subject to a uniform

TABLE of v/v_0.

r/a.	$z = 0$.	$z = \cdot 1c$.	$z = \cdot 2c$.	$z = \cdot 3c$.	$z = \cdot 4c$.	$z = \cdot 5c$.	$z = \cdot 6c$.	$z = \cdot 7c$.	$z = \cdot 8c$.	$z = \cdot 9c$.	$z = c$.
0	0	0	0	0	0	0	0	0	0	0	0
·2	0	·019704	·039235	·058350	·076671	·093656	·108646	·120983	·130136	·135953	·137645
·4	0	·039599	·078938	·117618	·154931	·189753	·220555	·245889	·264618	·276069	·279918
·6	0	·059762	·119340	·178387	·236094	·290682	·339273	·378990	·408066	·425706	·431612
1·0	0	·100258	·200717	·301772	·404584	·516879	·619041	·691974	·742997	·773388	·783489

TABLE of $\widehat{r\phi}/T_0$.

r/a.	$z = 0$.	$z = \cdot 1c$.	$z = \cdot 2c$.	$z = \cdot 3c$.	$z = \cdot 4c$.	$z = \cdot 5c$.	$z = \cdot 6c$.	$z = \cdot 7c$.	$z = \cdot 8c$.	$z = \cdot 9c$.	$z = c$.
0	0	0	0	0	0	0	0	0	0	0	0
·2	0	·000546	·001317	·002517	·004244	·006353	·008452	·010145	·011266	·011864	·012046
·4	0	·001821	·004540	·009126	·016259	·025423	·034559	·041589	·045928	·048101	·048743
·6	0	·002777	·007342	·016277	·032852	·057238	·081594	·098019	·106624	·110409	·111470
1·0	0	0	0	0	0	{ 0 { ·318310	·318310	·318310	·318310	·318310	·318310

TABLE of $\widehat{\phi z}/T_0$.

r/a.	$z = 0$.	$z = \cdot 1c$.	$z = \cdot 2c$.	$z = \cdot 3c$.	$z = \cdot 4c$.	$z = \cdot 5c$.	$z = \cdot 6c$.	$z = \cdot 7c$.	$z = \cdot 8c$.	$z = \cdot 9c$.	$z = c$.
0	0	0	0	0	0	0	0	0	0	0	0
·2	·19730	·19650	·19373	·18795	·17757	·16102	·13766	·10819	·07429	·03774	0
·4	·39638	·39519	·39096	·38147	·36288	·33076	·28300	·22179	·15169	·07682	0
·6	·59789	·59708	·59396	·58574	·56550	·52125	·44556	·34598	·23436	·11800	0
1·0	1·00230	1·00317	1·00658	1·01614	1·04654	∞	·84647	·61593	·40614	·20217	0

torsion over its whole length, the total couple applied being the same as in the present case. We find $T_0 = \mu\tau a = \pi T$ and $v_0 = \tau c a = \pi c T/\mu$ for the given example.

Tables are given on page 229.

Looking at these tables we see that, over the length free from external applied shear, the strains and stresses inside the cylinder are sensibly the same as what they would be on the hypothesis of a uniform torsion. Outside $z/c = \cdot 5$ the torsion couple diminishes, and the stresses diminish in consequence.

It is interesting to compare these results with those that we should have obtained if we had supposed the approximate results given on p. 223 to hold good in this case. Denoting by v', $\widehat{r\phi}'$, $\widehat{\phi z}'$ the values of the displacement and stresses calculated on this hypothesis, we have—

TABLE of v'/v_0.

r/a.	z/c = 0.	·1.	·2.	·3.	·4.	·5.	·6.	·7.	·8.	·9.	1.
0	0	0	0	0	0	0	0	0	0	0	0
·2	0	·020	·040	·080	·080	·100	·118	·132	·142	·148	·150
·4	0	·040	·080	·120	·160	·200	·236	·264	·284	·296	·300
·6	0	·060	·120	·180	·240	·300	·354	·396	·426	·444	·450
1·0	0	·100	·200	·300	·400	·500	·590	·660	·710	·740	·750

TABLE of $\widehat{r\phi}'/T_0$.

r/a.	z/c = 0.	·1.	·2.	·3.	·4.	·5.	·6.	·7.	·8.	·9.	1.
0	0	0	0	0	0	0	0	0	0	0	0
·2	0	0	0	0	0	{ 0 { ·012732	·012732	·012732	·012732	·012732	·012732
·4	0	0	0	0	0	{ 0 { ·050930	·050930	·050930	·050930	·050930	·050930
·6	0	0	0	0	0	{ 0 { ·114592	·114592	·114592	·114592	·114592	·114592
1·0	0	0	0	0	0	{ 0 { ·318310	·318310	·318310	·318310	·318310	·318310

TABLE of $\widehat{\phi z}'/T_0$.

r/a.	z/c = 0.	·1.	·2.	·3.	·4.	·5.	·6.	·7.	·8.	·9.	1
0	0	0	0	0	0	0	0	0	0	0	0
·2	·20	·20	·20	·20	·20	·20	·16	·12	·08	·04	0
·4	·40	·40	·40	·40	·40	·40	·32	·24	·16	·08	0
·6	·60	·60	·60	·60	·60	·60	·48	·36	·24	·12	0
1·0	1·00	1·00	1·00	1·00	1·00	1·00	·80	·60	·40	·20	0

§ 32. *Discussion of the Results.*

From the above tables we see that the radii in each cross-section do not remain straight lines, but assume distorted shapes, which are shown, on a very exaggerated scale, on Diagram 16, where, for each of the ten cross-sections, the curve of $(v - v')/v_0$, which indicates the deviation from the straight line in the distorted form of the radii, has been plotted. The variation from the straight line increases rapidly as we approach the region where the stress is applied, as can be seen from curves (1)–(4) on Diagram 16. On the other hand, towards the ends, the distortion remains fairly constant. The distorted radii meet the bounding circles at right angles when $\widehat{r\phi} = 0$ at the surface, but they meet it at a finite angle where $\widehat{r\phi} = \mathrm{T}$.

From the values of $\widehat{\phi z}$ and $\widehat{r\phi}$ we see that as soon as we get at all away from the ends the conditions that $\widehat{r\phi} = 0$, $\widehat{\phi z} = \mu\tau z$, $v = \tau r z$, which hold for uniform torsion, are very closely satisfied, and that, more generally, except where the abrupt change takes place in the shearing stress at the surface, the approximate expressions given in § 28 do not differ widely from the true expressions, the law that $\widehat{r\phi}$ varies as the square of the radius being, near the ends, tolerably well verified. It is to be noted also that, where the approximations would give a discontinuity in $\widehat{r\phi}$ *inside* the material (viz. at $z = \cdot 5c$), the true values are almost exactly the mean of the two discontinuous values obtained from the approximate formulæ assumed correct.

In like manner $\widehat{\phi z}$ is nearly the same as $\widehat{\phi z}'$, except near $z = \cdot 5c$, where, as we have seen, an infinite stress really occurs, of which the approximations give no hint.

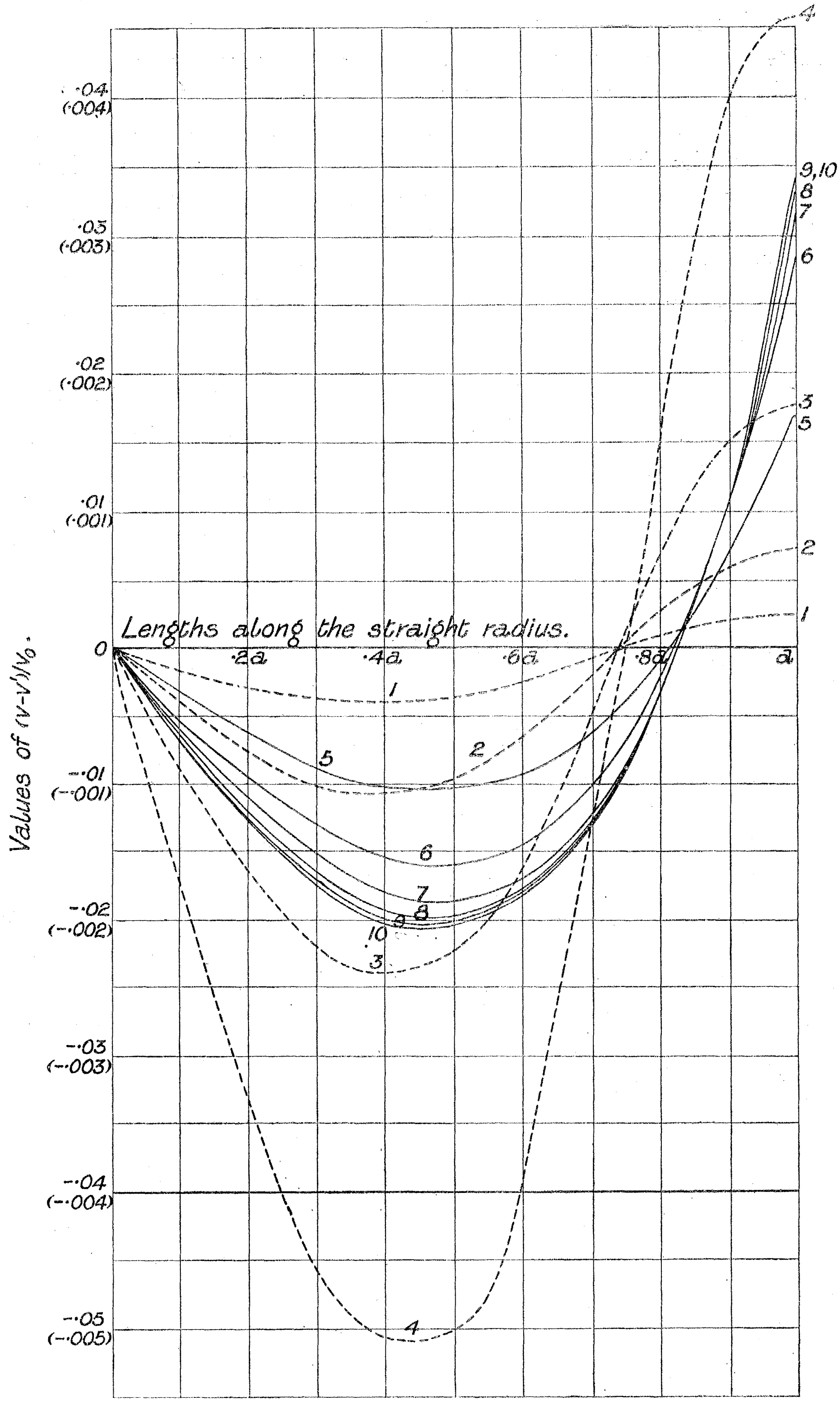
We note, however, that $\widehat{\phi z}$ does not strictly vary as r all over the section, being smaller than should be expected inside and larger at the boundary.

The theoretical result, that the stress is infinite where the transverse applied shear is discontinuous, throws much light on the case of a cylinder whose cross-section abruptly changes, as in fig. 1, with the difference that now the stress applied to the collar is transverse. We see that in such a case we should expect the material to give way at the points of sudden change. This conclusion is in accordance with practical experience, the tail ends of propeller shafts, for instance, breaking almost invariably in this manner.

§ 33. *General Conclusion.*

This example concludes the series of three which it was proposed to treat of. The object has been to obtain a clear idea of the effects of certain surface distributions of stress which come much nearer to the cases arising in practice than does the uniform distribution ordinarily taken.

Diagram 16.—Showing Distortion of a Radius originally Straight in the case of Torsion produced by applying Shearing Stress to the Curved Surface.



The curve corresponding to the section $z/c = n/10$ is numbered n. The first four curves have had the ordinates exaggerated in the ratio of 10 : 1. They are shown by the dotted lines, and to them refer the numbers in brackets.

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No doubt the cases treated involve somewhat arbitrary conditions, not strictly obtained in practice, but it appeared useful to ascertain how far they gave results diverging from those which would be found on the ordinary hypothesis of uniform extension or torsion.

This furnishes us with a test of how far we may accept DE SAINT-VENANT'S principle of "equipollent" systems of load for a bar whose length is gradually made smaller compared with its diameter. The results we have here obtained indicate that, as we go away from the points of application of the stress, a "uniform" solution is reached much sooner in the case of torsion than in that of either tension or pressure.

With regard to the arithmetic of the paper, the results have been as far as possible checked. It is believed that they are correct to the number of figures given, but owing to the slow convergence of certain of the series, accumulated errors may in some cases affect the last and even the second last figure. Even this, however, would not sensibly disturb the conclusions.

For the I-functions the tables in GRAY and MATHEW'S "Bessel's Functions" were used, but the range of the tables is so limited that a large number of these functions had to be independently calculated. The semi-convergent expansions were employed, the argument being large in each case.

My very best thanks are due to Professor EWING for his unfailing kindness in coming to my aid with suggestions and advice.

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